International Mathematical Forum, Vol. 13, 2018, no. 8, 351 - 356
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/imf.2018.8530

Group Inverse for a Class of Centrosymmetric Matrix

Mei Wang and Junqing Wang

Department of Mathematics, Tianjin Polytechnic University
Tianjin, China, 300385

Abstract

In this paper a class of centrosymmetric matrix is considered, its full rank decomposition is based on, and the group inverse for this decomposition is obtained, and the relationship between mother matrix and the group inverse for this class of centrosymmetric matrix is derived. The results show that the method can not only reduce the calculated amount and memory space, but also can not reduce the numerical accuracy; so extend the results of the related references and spread its application scope.

Keywords: centrosymmetric matrix; full rank decomposition; group inverse

1. Introduction

There are a lot of centrosymmetric matrix in linear system theory, numerical analysis and other fields, therefore, the study of symmetric matrix is of great importance. Full rank decomposition is the most commonly used method in matrix decomposition, and it plays important roles in finding the generalized inverse of the matrix. Scholars have studied the properties, invertibility and inverse problems of centrosymmetric matrices, but, based on the full rank decomposition of a class of special centrosymmetric matrices, this paper studies the group inverse of matrices on the basis of full rank decomposition and obtains the relationship between the group inverse and the mother matrix., and generalizes and verifies the conclusions in the literature.

Definition 1 If $M=\left(m_{ij}\right)\in\mathbb{C}^{n\times n}$ satisfies $m_{ij}=m_{n+1-i,n+1-j}, i,j=1,2,\cdots,n$, then M is called the centrosymmetric matrix.

Lemma 1[3] According to the definition 1, we have

(1) The $2m$- order centrosymmetric matrix M can be expressed as

$$M=\begin{pmatrix} A & BJ \\ JB & JAJ \end{pmatrix},$$

(2) The $2m+1$- order centrosymmetric matrix M can be expressed as

$$M=\begin{pmatrix} A & u & BJ \\ v^T & \alpha & v^T J \\ JB & Ju & JAJ \end{pmatrix},$$

Where A,B is an m-order square matrix, u,v are m-dimensional column vector, α is an arbitrary number, J is a m-order square matrix with sub-diagonal elements of 1 and the remaining elements of 0.

In particular, according to the lemma 1, if $A=B$, u,v are m-dimensional zero vector, $\alpha=0$, then

(1) The $2m$- order centrosymmetric matrix M can be expressed as

$$M=\begin{pmatrix} A & AJ \\ JA & JAJ \end{pmatrix},$$

(2) The $2m+1$- order centrosymmetric matrix M can be expressed as

$$M=\begin{pmatrix} A & \mathrm{O} & AJ \\ \mathrm{O}^T & 0 & \mathrm{O}^T \\ JA & \mathrm{O} & JAJ \end{pmatrix},$$

The matrix M can be regarded as the extended matrix of matrix A, and A is its mother matrix.

Lemma 2[2] Let the full rank decomposition of $A\in\mathbb{C}^{m\times m}$ is $A=FG$, where $F\in\mathbb{C}^{m\times r}$ is column full rank decomposition, $G\in\mathbb{C}^{r\times n}$ is row full rank matrix, and

$rankA = rankF = rankG = r\left(r > 0\right)$, then

$$M = \begin{pmatrix} A & AJ \\ JA & JAJ \end{pmatrix} = \begin{pmatrix} F \\ JF \end{pmatrix} \begin{pmatrix} G & GJ \end{pmatrix}$$

is the full rank decomposition of the $2m$- order centrosymmetric matrix M .

Lemma 3[2] Let the full rank decomposition of $A \in \mathbb{C}^{m\times m}$ is $A = FG$, where $F \in \mathbb{C}^{m\times r}$ is column full rank decomposition, $G \in \mathbb{C}^{r\times n}$ is row full rank matrix, and $rankA = rankF = rankG = r\left(r > 0\right)$, then

$$M = \begin{pmatrix} A & \mathrm{O} & AJ \\ \mathrm{O}^T & 0 & \mathrm{O}^T \\ JA & \mathrm{O} & JAJ \end{pmatrix} = \begin{pmatrix} F \\ \mathrm{O}^T \\ JF \end{pmatrix} \begin{pmatrix} G & \mathrm{O} & GJ \end{pmatrix}$$

is the full rank decomposition of the $2m+1$- order centrosymmetric matrix M .

2. Theorem

Lemma 4[1] Let the full rank decomposition of $A \in \mathbb{C}^{m\times m}$ is $A = FG$, $A^{\#}$ exists if and only if FG is reversible, then

$$A^{\#} = F\left[\left(GF\right)^{-1}\right]^2 G .$$

Proof: Let $X = F\left[\left(GF\right)^{-1}\right]^2 G$, the following verification of X satisfies the three conditions in the group inverse definition:

$$AXA = FGF\left[\left(GF\right)^{-1}\right]^2 GFG = FGFF^{-1}G^{-1}F^{-1}G^{-1}GFG = FG = A,$$

$$XAX = F\left[\left(GF\right)^{-1}\right]^2 GFGF\left[\left(GF\right)^{-1}\right]^2 G = FF^{-1}G^{-1}F^{-1}G^{-1}GFGFF^{-1}G^{-1}F^{-1}G^{-1}G$$

$$= FF^{-1}G^{-1}F^{-1}G^{-1}G = F\left[\left(GF\right)^{-1}\right]^2 G = X,$$

$$AX = FGF\left[\left(GF\right)^{-1}\right]^2 G = FGFF^{-1}G^{-1}F^{-1}G^{-1}G = I,$$

$$XA = F\left[\left(GF\right)^{-1}\right]^2 GFG = FF^{-1}G^{-1}F^{-1}G^{-1}GFG = I,$$

so X is the group inverse of $A = FG$.

Theorem 1 Let the full rank decomposition of $A\in\mathbb{C}^{m\times m}$ is $A=FG$, where $F\in\mathbb{C}^{m\times r}$ is column full rank decomposition, $G\in\mathbb{C}^{r\times n}$ is row full rank matrix, and $rankA=rankF=rankG=r\left(r>0\right)$, then the group inverse of the $2m$- order centrosymmetric matrix $M=\begin{pmatrix} A & AJ \\ JA & JAJ \end{pmatrix}$ is: $M^{\#}=\frac{1}{4}\begin{pmatrix} A^{\#} & A^{\#}J \\ JA^{\#} & JA^{\#}J \end{pmatrix}$.

Proof: Let the full rank decomposition of $A\in\mathbb{C}^{m\times m}$ is $A=FG$, by theorem, we have $\begin{pmatrix} F \\ JF \end{pmatrix}\begin{pmatrix} G & GJ \end{pmatrix}$ is the full rank decomposition of $M=\begin{pmatrix} A & AJ \\ JA & JAJ \end{pmatrix}$,

therefore:

$$\begin{aligned} M^{\#} &= \begin{pmatrix} F \\ JF \end{pmatrix}\left[\left(\begin{pmatrix} G & GJ \end{pmatrix}\begin{pmatrix} F \\ JF \end{pmatrix}\right)^{-1}\right]^2 \begin{pmatrix} G & GJ \end{pmatrix} \\ &= \begin{pmatrix} F \\ JF \end{pmatrix}\left[\left(2GF\right)^{-1}\right]^2 \begin{pmatrix} G & GJ \end{pmatrix} = \frac{1}{4}\begin{pmatrix} F \\ JF \end{pmatrix}\left[\left(GF\right)^{-1}\right]^2 \begin{pmatrix} G & GJ \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} F\left[\left(GF\right)^{-1}\right]^2 G & F\left[\left(GF\right)^{-1}\right]^2 GJ \\ JF\left[\left(GF\right)^{-1}\right]^2 G & JF\left[\left(GF\right)^{-1}\right]^2 GJ \end{pmatrix} = \frac{1}{4}\begin{pmatrix} A^{\#} & A^{\#}J \\ JA^{\#} & JA^{\#}J \end{pmatrix}. \end{aligned}$$

Theorem 2 Let the full rank decomposition of $A\in\mathbb{C}^{m\times m}$ is $A=FG$, where $F\in\mathbb{C}^{m\times r}$ is column full rank decomposition, $G\in\mathbb{C}^{r\times n}$ is row full rank matrix, and $rankA=rankF=rankG=r\left(r>0\right)$, then the group inverse of the $2m+1$- order centrosymmetric matrix $M=\begin{pmatrix} A & \mathrm{O} & AJ \\ \mathrm{O}^T & 0 & \mathrm{O}^T \\ JA & \mathrm{O} & JAJ \end{pmatrix}$ is: $M^{\#}=\frac{1}{4}\begin{pmatrix} A^{\#} & \mathrm{O} & A^{\#}J \\ \mathrm{O}^T & 0 & \mathrm{O}^T \\ JA^{\#} & \mathrm{O} & JA^{\#}J \end{pmatrix}$.

Proof: Let the full rank decomposition of $A\in\mathbb{C}^{m\times m}$ is $A=FG$, by theorem, we have $\begin{pmatrix} F \\ O^T \\ JF \end{pmatrix}\begin{pmatrix} G & O & GJ \end{pmatrix}$ is the full rank decomposition of $M=\begin{pmatrix} A & O & AJ \\ O^T & 0 & O^T \\ JA & O & JAJ \end{pmatrix}$,

therefore:

$$M^{\#}=\begin{pmatrix} F \\ O^T \\ JF \end{pmatrix}\left[\left(\begin{pmatrix} G & O & GJ \end{pmatrix}\begin{pmatrix} F \\ O^T \\ JF \end{pmatrix}\right)^{-1}\right]^2\begin{pmatrix} G & O & GJ \end{pmatrix}$$

$$=\begin{pmatrix} F \\ O^T \\ JF \end{pmatrix}\left[(2GF)^{-1}\right]^2\begin{pmatrix} G & O & GJ \end{pmatrix}=\frac{1}{4}\begin{pmatrix} F \\ O^T \\ JF \end{pmatrix}\left[(GF)^{-1}\right]^2\begin{pmatrix} G & O & GJ \end{pmatrix}$$

$$=\frac{1}{4}\begin{pmatrix} F\left[(GF)^{-1}\right]^2 G & O & F\left[(GF)^{-1}\right]^2 GJ \\ O^T & 0 & O^T \\ JF\left[(GF)^{-1}\right]^2 G & O & JF\left[(GF)^{-1}\right]^2 GJ \end{pmatrix}=\frac{1}{4}\begin{pmatrix} A^{\#} & O & A^{\#}J \\ O^T & 0 & O^T \\ JA^{\#} & O & JA^{\#}J \end{pmatrix}.$$

3. Conclusion

To sum up, for the full rank decomposition of the centrosymmetric matrix, it is found that the full rank decomposition of the mother matrix A can be performed first, and then the full rank decomposition can be easily calculated by using the conclusions in the literature. For the group inverse of the centrosymmetric matrix, through research, we can find the group inverse of its mother matrix and then use the theorem of this paper to calculate the result. Therefore, it is very easy to calculate the group inverse of the centrosymmetric matrix through the inverse of the mother matrix. The calculation method not only reduces the amount of calculation, but also it does not lose accuracy.

References

[1] Changjiang Pu, Yimin Wei, *The Symbolic Pattern of Generalized Inverse*, Beijing: Science Press, 2014.

[2] Wei Guo, Full Rank Decomposition and Moore-Penrose Inverses of Completely Symmetric Matrices, *Journal of Sichuan Normal University (Natural Science)*, **32** (2009), no. 4, 454-457.

[3] Zhong Xu, Quan Lu, The Inverse and Generalized Inverses of Centrosymmetric and Anti-centrosymmetric Symmetric Matrices, *Mathematical Bulletin*, **3** (1998), 37-42.

[4] JingPin Huang, The Drazin Inverse of a Centrosymmetric Matrix, *Journal of Chongqing Teachers College* (Natural Science Edition), **16** (1999), no. 1, 64-68.

Received: June 3, 2018; Published: June 29, 2018