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Logarithm of the Kernel Function

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Abstract

In this note we study the logarithm of the kernel function and related functions.

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1 Introduction and Preliminary Results

A square-free number is a number without square factors, a product of different primes. The first few terms of the integer sequence of square-free numbers are

$$1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, \ldots$$

Let us consider the prime factorization of a positive integer $n \geq 2$

$$n = q_1^{s_1} q_2^{s_2} \cdots q_t^{s_t}$$

where $q_1, q_2, \ldots q_t$ are the different primes in the prime factorization.

We have the following two arithmetical functions

$$u(n) = q_1 q_2 \cdots q_t$$

The arithmetical function $u(n)$ is well-known in the literature, it is called kernel of n, radical of n, etc. Note that $u(n)$ is the largest squarefree number that divides n. There exist many papers dedicated to this function.

$$v(n) = \frac{n}{u(n)} = q_1^{s_1-1} q_2^{s_2-1} \cdots q_t^{s_t-1}$$

We call $v(n)$ the remainder of n. Note that $v(n) = 1$ if and only if n is a square-free.

In this note we obtain asymptotic formulae for the following sums.

$$\sum_{1\leq i\leq n} \log u(i), \qquad \sum_{1\leq i\leq n} \log v(i)$$

$$\sum_{1\leq i\leq n} \frac{\log u(i)}{\log n}, \qquad \sum_{1\leq i\leq n} \frac{\log v(i)}{\log n}$$

As usual, $\lfloor x\rfloor$ denotes the integer part of x, $\{x\} = x - \lfloor x\rfloor$ denotes the fractional part of x, p denotes a positive prime, the simbol $\sum_p$ mean that the sum run on all positive primes.

We need the following well-known lemmas.

Lemma 1.1 *The following asymptotic formula holds*

$$\sum_{1\leq i\leq n} \frac{1}{i} = \log n + \gamma + o(1)$$

where $\gamma = 0,5772156649\ldots$ is called Euler's constant (see [2, chapter 1]).

Lemma 1.2 *The following asymptotic formula holds*

$$\sum_{2\leq p\leq x} \frac{\log p}{p} = \log x + C + o(1)$$

where

$$C = -\gamma - \sum_p \frac{\log p}{p(p-1)} = -1,3325822757\ldots$$

(See [2, chapter 2])

Lemma 1.3 *(Prime number theorem) The following asymptotic formula holds*

$$\sum_{2\leq p\leq x} \log p = x + o(x)$$

Lemma 1.4 *The following asymptotic formula holds*

$$\sum_{1\leq i\leq n} \log i = n\log n - n + o(n)$$

Proof. It is a weak consequence of the Stirling's formula $n! \sim \frac{\sqrt{2\pi}n^n\sqrt{n}}{e^n}$.

2 Main Results

Theorem 2.1 *The following asymptotic formulae hold*

$$\sum_{1\le i\le n} \log u(i) = n\log n - (1+A)n + o(n) \tag{1}$$

$$\sum_{1\le i\le n} \log v(i) = An + o(n) \tag{2}$$

where $A = \sum_p \frac{\log p}{p(p-1)} = 0,7553\ldots$.

Proof. Equation (2) is an immediate consequence of equation (1) and Lemma 1.4, since $u(i)v(i) = i$. Therefore we shall prove equation (1). Note that if j is an arbitrary but fixed positive integer and the prime p satisfies the inequality $\frac{n}{j+1} < p \le \frac{n}{j}$ then $\left\lfloor \frac{n}{p} \right\rfloor = j$. Let $\epsilon > 0$, we choose the fixed positive integer s such that

$$\frac{1}{s} < \epsilon \tag{3}$$

and in the equation (see Lemma 1.1)

$$\sum_{j=1}^{s} \frac{1}{j} = \log s + \gamma + o_1(1)$$

we have

$$|o_1(1)| < \epsilon$$

We have

$$\sum_{1\le i\le n} \log u(i) = \sum_{2\le p\le n} \log p \left\lfloor \frac{n}{p} \right\rfloor = \left(\sum_{j=1}^{s-1} j \sum_{\frac{n}{j+1}<p\le\frac{n}{j}} \log p \right) + \sum_{2\le p\le \frac{n}{s}} \log p \left\lfloor \frac{n}{p} \right\rfloor \tag{4}$$

We have by Lemma 1.3 and Lemma 1.1

$$\begin{aligned} \sum_{j=1}^{s-1} j \sum_{\frac{n}{j+1}<p\le\frac{n}{j}} \log p &= \sum_{j=1}^{s-1} j \left(\frac{n}{j} - \frac{n}{j+1} + o(n) \right) = \sum_{j=1}^{s-1} j \left(\frac{n}{j} - \frac{n}{j+1} \right) \\ + \quad o(n) &= n \sum_{j=1}^{s} \frac{1}{j} - n + o(n) = n\left(\log s + \gamma + o_1(1)\right) - n + o_2(1)n \end{aligned} \tag{5}$$

On the other hand, we have by Lemma 1.2

$$\begin{aligned} \sum_{2\le p\le \frac{n}{s}} \log p \left\lfloor \frac{n}{p} \right\rfloor &= n \sum_{2\le p\le \frac{n}{s}} \frac{\log p}{p} - \sum_{2\le p\le \frac{n}{s}} \log p \left\{ \frac{n}{p} \right\} \\ &= n\left(\log\left(\frac{n}{s}\right) + C + o_3(1) \right) - \sum_{2\le p\le \frac{n}{s}} \log p \left\{ \frac{n}{p} \right\} \end{aligned} \tag{6}$$

where by Lemma 1.3 and equation (3)

$$0 \leq \sum_{2 \leq p \leq \frac{n}{s}} \log p \left\{ \frac{n}{p} \right\} \leq \sum_{2 \leq p \leq \frac{n}{s}} \log p = \left(\frac{1}{s} + o_4(1) \right) n \leq 2\epsilon n$$

That is

$$0 \leq \frac{\sum_{2 \leq p \leq \frac{n}{s}} \log p \left\{ \frac{n}{p} \right\}}{n} \leq 2\epsilon \tag{7}$$

We have choose n_0 such that if $n \geq n_0$ then $|o_2(1)| < \epsilon$, $|o_3(1)| < \epsilon$ and $|o_4(1)| < \epsilon$.

Substituting equations (5) and (6) into equation (4) we find that

$$\sum_{1 \leq i \leq n} \log u(i) = n \log n - \left(1 + \sum_p \frac{\log p}{p(p-1)} \right) n + o_1(1)n + o_2(1)n + o_3(1)n - \frac{\sum_{2 \leq p \leq \frac{n}{s}} \log p \left\{ \frac{n}{p} \right\}}{n} n$$

Therefore (see equation (7))

$$\left| \frac{\sum_{1 \leq i \leq n} \log u(i) - n \log n + \left(1 + \sum_p \frac{\log p}{p(p-1)} \right) n}{n} \right| \leq |o_1(1)| + |o_2(1)| + |o_3(1)| + \frac{\sum_{2 \leq p \leq \frac{n}{s}} \log p \left\{ \frac{n}{p} \right\}}{n} \leq 5\epsilon \qquad (n \geq n_0)$$

That is

$$\frac{\sum_{1 \leq i \leq n} \log u(i) - n \log n + \left(1 + \sum_p \frac{\log p}{p(p-1)} \right) n}{n} = o(1)$$

That is, equation (1), since ϵ can be arbitrarily small. The theorem is proved.

An immediate consequence of Stirling's formula (see above) is the well-known limit

$$\frac{\sqrt[n]{n!}}{n} \to \frac{1}{e}$$

Note that

$$n! = \prod_{i=1}^{n} i = \prod_{i=1}^{n} (u(i)v(i)) = \left(\prod_{i=1}^{n} u(i) \right) \left(\prod_{i=1}^{n} v(i) \right)$$

An immediate consequence of Theorem 2.1 is the following corollary.

Corollary 2.2 *The following limits hold*

$$\frac{\sqrt[n]{\prod_{i=1}^{n} u(i)}}{n} \to \frac{1}{e^{1+A}} \qquad \sqrt[n]{\prod_{i=1}^{n} v(i)} \to e^{A}$$

Theorem 2.3 *The following asymptotic formulae hold*

$$\sum_{2\leq n\leq x} \frac{\log v(n)}{\log n} = A\frac{x}{\log x} + o\left(\frac{x}{\log x}\right) \tag{8}$$

$$\sum_{2\leq n\leq x} \frac{\log u(n)}{\log n} = x - A\frac{x}{\log x} + o\left(\frac{x}{\log x}\right) \tag{9}$$

Proof. Note that

$$\begin{aligned} x + o\left(\frac{x}{\log x}\right) &= \lfloor x \rfloor - 1 = \sum_{2\leq n\leq x} \frac{\log n}{\log n} = \sum_{2\leq n\leq x} \frac{\log(u(n)v(n))}{\log n} \\ &= \sum_{2\leq n\leq x} \frac{\log u(n)}{\log n} + \sum_{2\leq n\leq x} \frac{\log v(n)}{\log n} \end{aligned} \tag{10}$$

Therefore (9) is an immediate consequence of (8) and (10). The proof of (8) is by partial summation (see [3, chapter XXII]).

We have by Theorem 2.1

$$\sum_{2\leq n\leq x} \log v(n) = Ax + o(x)$$

Consequently, if we use the function $f(x) = \frac{1}{\log x}$ then

$$\begin{aligned} \sum_{2\leq n\leq x} \frac{\log v(n)}{\log n} &= (Ax + o(x))\, f(x) - \int_2^x (At + o(t))\, f'(t)\ dt \\ &= A\frac{x}{\log x} + o\left(\frac{x}{\log x}\right) + A\int_2^x \frac{1}{\log^2 t}\ dt + \int_2^x o(1)\frac{1}{\log^2 t}\ dt \\ &= A\frac{x}{\log x} + o\left(\frac{x}{\log x}\right) \end{aligned}$$

That is, equation (8). The theorem is proved.

3 An Application to Sums of Fractional Parts

Dirichlet (1849) proved the asymptotic formula

$$\sum_{1\leq k\leq n}\left\{\frac{n}{k}\right\}=(1-\gamma)n+O\left(\sqrt{n}\right)$$

De la Vallée Poussin [1] proved the asymptotic formula

$$\sum_{1\leq p\leq n}\left\{\frac{n}{p}\right\}=(1-\gamma)\frac{n}{\log n}+o\left(\frac{n}{\log n}\right)$$

Note that an immediate consequence of Lemma 1.2, (1) and (4) is the asymptotic formula

$$\sum_{1\leq p\leq n}\log p\left\{\frac{n}{p}\right\}=(1-\gamma)n+o(n)$$

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