International Mathematical Forum, Vol. 13, 2018, no. 6, 293 - 302
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/imf.2018.828

A Remark on Solutions of Lamé Equations with Elliptic Potentials

U. N. Bassey[1] **and U. S. Idiong**

Department of Mathematics
University of Ibadan, Ibadan, Nigeria

Copyright © 2018 U. N. Bassey and U. S. Idiong. This article is distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

There are distinct equations in the literature referred to as the Lamé equations. The equations are the Lamé equation in the algebraic form, Lamé equation in the Jacobi form, Lamé equation in the Weierstrass form, among others. It is well known that Lamé equation may be linked to one another through suitable variable transformations. The link between their solutions has not yet been investigated. The purpose of the present note is to establish a link between the solution of Lamé equation in its Weierstrass form which extends to an elliptic function and the solution of Lamé equation in the Jacobi form.

Mathematics Subject Classification: Primary 97I80 and Secondary 34Mxx

Keywords: Elliptic functions, Elliptic potentials, Lamé equation in the Jacobi form, Lamé equation in the Weierstrass form

1 Introduction

The Lamé equation in the Weierstrass form is given by

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2} - n(n+1)\wp(z; g_2, g_3) - B\right\}\Psi(z) = 0, \tag{1.1}$$

where $\wp(z; g_2, g_3)$ is the Weierstrass elliptic $\wp$-function and g_2 and g_3 are invariant parameters satisfying the equation $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$, and B is the

[1]Corresponding author

accessory parameter which plays the role of the eigenvalue. In this classical setting, n is a positive integer. The linearly independent solutions of equation (1.1), when $n = 1$, are given in terms of Weierstrass sigma $\sigma(z)$ and zeta $\zeta(z)$ elliptic functions as

$$\Psi^{\pm}(z) = \frac{\sigma(z \pm \varepsilon)}{\sigma(z)\sigma(\varepsilon)} \exp\left(\mp z\zeta(\varepsilon)\right). \tag{1.2}$$

Here, ε is a parameter connected with B by means of transcendental equation $\wp(\varepsilon) = B$. In Bassey and Idiong [4], some exactly solvable potentials were constructed giving rise to solutions of equation (1.1) which could be written in terms of some classical orthogonal polynomials.

The Jacobi form of Lamé equation is given by

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}u^2} - n(n+1)k^2\mathrm{sn}^2(u) - E\right\}\Psi(u) = 0. \tag{1.3}$$

Here, $k \in (0,1)$ is the modulus parameter, $\mathrm{sn}(u)$ is the Jacobi amplitude sine function and E is the eigenvalue. In what follows, $H(u), \Theta(u)$ and $Z(u)$ denote the Jacobi Eta, Theta and Zeta functions, respectively. The linearly independent solutions of equation (1.3) are given as

$$\Psi^{\pm}(u) = \frac{H(u \pm \alpha)}{\Theta(u)\Theta(\alpha)} \exp\left(\mp uZ(\alpha)\right), \tag{1.4}$$

where $\mathrm{dn}^2\alpha = E - k^2$ and $\mathrm{dn}(\cdot)$ is the delta amplitude function (see Ince [7], p. 395).

As it is well known (cf: Whittaker and Watson [8], p.555), the equations (1.1) and (1.3) may be linked to one another through variable transformations. In section 3 of the present paper, we establish a link between the solutions of these equations presented in (1.2) and (1.4). In section 2, we present the necessary preliminaries that are required.

2 Preliminaries

In this section, we set down necessary notations and preliminary definitions concerning elliptic functions (and their relationship with each other) which are required in the sequel. We present, for the sake of completeness, a transformation of the Lamé equation from the Weierstrass form to the Jacobi form. Definition 2.1 below is found in Whittaker and Watson ([8], § 21.11, p.463).

2.1 Definition. Let ω_j $(j = 1, 2, 3)$ be the elliptic periods of the Weierstrass elliptic functions mentioned in section 1 above and let $\tau = \frac{\omega_2}{\omega_1}$ be a complex constant with the imaginary part, $\Im(\tau) > 0$. We define the parameter $q := e^{i\pi\tau}$ $(i = \sqrt{-1})$ so that $|q| < 1$. Then, the function $\vartheta(z, q)$ defined by the series

$$\vartheta(z, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niz} \tag{2.1}$$

is called the *qua function* of variable z.

If $M \in \mathbb{R}^+$, then when $|z| \leq M$, we have

$$|q^{n^2} e^{inz}| \leq |q|^{n^2} e^{2nM} < e^{2nM},$$

n being a positive integer. Equation (2.1) can be re-written as

$$\vartheta(z, q) = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nz,$$

and

$$\begin{aligned} \vartheta(z + \pi, q) &= 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2n(z + \pi) \\ &= 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2n(z + \pi) \\ &= 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nz \\ &= \vartheta(z, q). \end{aligned} \tag{2.2}$$

Furthermore, we have

$$\begin{aligned} \vartheta(z + \pi\tau, q) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2ni(z+\pi\tau)} \\ &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2inz} e^{2\pi\tau} \\ &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} q^{2n} e^{2inz} \\ &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2+2n} e^{2inz} \\ &= \sum_{n=-\infty}^{\infty} (-1)^n q^{-1} q^{n^2+2n+1} e^{-i2z} e^{2i(n+1)z} \\ &= q^{-1} e^{-i2z} \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+1)^2} e^{2i(n+1)z} \\ &= q^{-1} e^{-i2z} \vartheta(z, q). \end{aligned} \tag{2.3}$$

Thus, $\vartheta(z,q)$ is a quasi-periodic function of z. In equations (2.1) and (2.3) above, 1 and $q^{-1}e^{-i2z}$ are called periodic factors associated with the periods π and $\pi\tau$ respectively. It is customary to write $\vartheta_4(z,q)$ in place of $\vartheta(z,q)$ while the other three qua functions are ϑ_3, ϑ_2 and ϑ_1, and as we see below all the ϑ_i $(i=1,2,3)$ may be expressed in relation to ϑ_4. Now, ϑ_3 is related to ϑ_4 as follows.

$$\begin{aligned}
\vartheta_3(z,q) &= \vartheta_4(z+\frac{\pi}{2},q) \\
&= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} e^{2ni(z+\frac{\pi}{2})} \\
&= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} e^{2niz} e^{in\pi} \\
&= \sum_{n=-\infty}^{\infty}(-1)^{2n} q^{n^2} e^{2niz} \\
&= \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niz} \\
&= 1+2\sum_{n=1}^{\infty} q^{n^2}\cos 2nz. \qquad (2.4)
\end{aligned}$$

Let $G=\prod_{n=1}^{\infty}(1-q^{2n})$. Then, in its product form, ϑ_3 may be written as

$$\vartheta_3(z,q)=G\prod_{n=1}^{\infty}(1+2q^{2n-1}\cos 2z+q^{4n-2}).$$

Also, $\vartheta_1(z,q)$ is related to $\vartheta_4(z,q)$ as follows.

$$\begin{aligned}
\vartheta_1(z,q) &= -ie^{i(z+\frac{\pi\tau}{4})}\vartheta_4(z+\frac{\pi\tau}{2},q) \\
&= -ie^{iz}e^{\frac{\pi\tau}{4})}\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} e^{2ni(z+\frac{\pi\tau}{2})} \\
&= -ie^{iz}q^{\frac{1}{4}}\sum_{n=-\infty}^{\infty}(-1)^n e^{2niz} q^{n^2+n} \\
&= -i\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2+n+\frac{1}{4}} e^{(2n+1)iz} \\
&= 2\sum_{n=-\infty}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2} e^{(2n+1)iz} \\
&= 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin(2n+1)z. \qquad (2.5)
\end{aligned}$$

The product form of $\vartheta_1(z,q)$ is expressed as

$$\vartheta_1(z,q) = 2Gq^{\frac{1}{4}} \sin z \prod_{n=1}^{\infty}(1 - 2q^{2n-1}\cos 2z + q^{4n}).$$

Then we can write $\vartheta_2(z,q)$ as

$$\begin{aligned}
\vartheta_2(z,q) &= \vartheta_1(z + \frac{\pi}{2}, q) \\
&= 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2} \sin(2n+1)(z+\frac{\pi}{2}) \\
&= 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2} \sin((2n+1)z + n\pi + \frac{\pi}{2}) \\
&= 2\sum_{n=0}^{\infty}(-1)^n \cos(2n+1)z. \qquad (2.6)
\end{aligned}$$

The product form of $\vartheta_2(z,q)$ is given as

$$\vartheta_2(z,q) = 2Gq^{\frac{1}{4}} \cos z \prod_{n=1}^{\infty}(1 + 2q^{2n-1}\cos 2z + q^{4n}).$$

The function ϑ_4 also has its infinite product form given as

$$\vartheta_4(z,q) = G\prod_{n=1}^{\infty}(1 - 2q^{2n-1}\cos 2z + q^{4n-2}).$$

It is now convenient for us to consider the Jacobi elliptic functions in terms of qua functions.

2.2 Definition (Whittaker and Watson [8],§21.69, p.479)**.** The Jacobi Theta function, $\Theta(u)$, is defined, in terms of qua function, by

$$\Theta(u) := \vartheta_4(u\vartheta_3^{-2}(0)|\tau). \qquad (2.7)$$

It is a periodic function with periods $2\mathbf{K}$ and $2i\mathbf{K}'$. Also,

$$\Theta(u+\mathbf{K}) := \vartheta_2(u\vartheta_3^{-2}(0)|\tau). \qquad (2.8)$$

Here and hereafter, $\mathbf{K} \equiv \mathbf{K}(k^2)$ and $\mathbf{K}' \equiv \mathbf{K}(k'^2)$ are, respectively, the complete elliptic integrals of the first kind and its complement, given as

$$\mathbf{K}(k^2) := \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}};$$

$$\mathbf{K}(k'^2) := \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k'^2t^2)}},$$

where k is the modulus and k' is the complementary modulus which obey the equation $k^2 + k'^2 = 1$ (see Akhiezer[2], p.78). The Jacobi eta function $H(u)$ is defined as

$$H(u) := -iq^{-\frac{1}{4}} e^{i\frac{\pi u}{2\mathbf{K}}} \Theta(u + i\mathbf{K}') = \vartheta_1(u\vartheta_3^{-2}(0)|\tau). \tag{2.9}$$

and

$$H(u + \mathbf{K}) := \vartheta_2(u\vartheta_3^{-2}(0)|\tau). \tag{2.10}$$

2.3 Definition (Baker [3], Chapter IX, p.74)**.** The Jacobi elliptic sine amplitude, cosine amplitude and delta amplitude functions are respectively defined, in terms of Theta and Eta functions, as

$$\mathrm{sn}(u,k) := \frac{1}{\sqrt{k}}\frac{H(u)}{\Theta(u)}, \qquad k \in (0,1) \tag{2.11}$$

$$\mathrm{cn}(u,k) := \sqrt{\frac{k'}{k}}\frac{H(u+\mathbf{K})}{\Theta(u)} \tag{2.12}$$

$$\mathrm{dn}(u,k) := \sqrt{k'}\frac{\Theta(u+\mathbf{K})}{\Theta(u)} \tag{2.13}$$

2.4 Definition (Byrd [5],§1035.01, p.310)**.** The Weierstrass elliptic $\wp$-function is defined, in terms of the qua function, as

$$\wp(z|\omega_1,\omega_2) := \frac{1}{12\omega_1^2}\frac{\vartheta_1'''(0)}{\vartheta_1'(0)} - \frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\ln \vartheta_1(\frac{z}{2\omega_1})\right) \tag{2.14}$$

2.5 Definition (Whittaker and Watson [8],§21.43, p.473)**.** The Weierstrass sigma function is defined, in terms of qua function, as

$$\sigma(z|\omega_1,\omega_2) = \frac{2\omega_1}{\pi}\exp\left(\frac{\eta_1 z^2}{2\omega_1}\right)\cdot\frac{1}{2}q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1-q^{2n})^{-3}\vartheta\left(\frac{\pi z}{2\omega_1}|\tau\right). \tag{2.15}$$

Here, $\tau = \frac{\omega_2}{\omega_1} \in \mathbb{C}$ such that $\Im(\tau) > 0$ and $\eta_1 = -\frac{\pi^2}{12\omega_1}\frac{\vartheta_1'''(0)}{\vartheta_1'(0)}$.

To complete this section, we show that Lamé equation in the Weierstrass form may be linked to the Jacobi form through variable transformation. Now, we know (Whittaker and Watson [8], §22.351, p.505) that

$$\wp(z; g_2, g_3) = e_3 + (e_1 - e_3)\mathrm{ns}^2(z\sqrt{e_1 - e_3}\ ). \tag{2.16}$$

By setting $u = z\sqrt{e_1 - e_3}$, we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2} = (e_1 - e_3)\frac{\mathrm{d}^2}{\mathrm{d}u^2}. \tag{2.17}$$

The Lamé equation (1.1), by expressions (2.17) and (2.16), now transforms into the equation

$$\left\{(e_1 - e_3)\frac{\mathrm{d}^2}{\mathrm{d}u^2} - \left[n(n+1)\left(e_3 + (e_1 - e_3)\mathrm{ns}^2(u)\right) + B\right]\right\}\Psi(u) = 0. \tag{2.18}$$

The last equation may be re-written as

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}u^2} - n(n+1)\mathrm{ns}^2(u) - \frac{n(n+1)e_3 + B}{e_1 - e_3}\right\}\Psi(u) = 0.$$

We know also (Whittaker and Watson [8], §22.34, p.507) that

$$\mathrm{sn}(u + i\mathbf{K}') = k^{-1}\mathrm{ns}u \Longrightarrow \mathrm{ns}^2 u = k^2\mathrm{sn}^2(u + i\mathbf{K}').$$

Thus, we have

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}u^2} - n(n+1)k^2\mathrm{sn}^2(u + i\mathbf{K}') - E\right\}\Psi(u) = 0, \tag{2.19}$$

where $E = \dfrac{n(n+1)e_3 + B}{e_1 - e_3}$. Since $\mathrm{sn}(u+i\mathbf{K}') = \mathrm{sn}(u)$, then equation (2.19) becomes

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}u^2} - n(n+1)k^2\mathrm{sn}^2(u) - E\right\}\Psi(u) = 0.$$

The variable transformation of Lamé equation in the Jacobi form to the equation in the Weierstrass form is obtained in Erdelyi et.al.([6], §15.2, pp.55-56).

We are now ready to present the main result of this paper.

3 Main Result

The aim of this section is to establish Theorem 3.1 below.

3.1 Theorem. The solution of Lamé equation in the Weierstrass form is equal to the solution of the equation in the Jacobi form multiplied by an elliptic function.

Proof. We know (Abramowitz and Stegun [1],§18.10.7-8, p. 650) that

$$\zeta(z) = \frac{\eta z}{\omega} + \frac{\pi}{\omega}\frac{\vartheta_1'(\nu)}{\vartheta_1(\nu)}, \tag{3.1}$$

$$\sigma(z) = \frac{2\omega}{\pi}\exp\left(\frac{\eta z^2}{2\omega}\right)\frac{\vartheta_1(\nu)}{\vartheta_1'(0)}. \tag{3.2}$$

Here and hereafter, $\nu = \dfrac{\pi z}{2\omega}$ and

$$\begin{aligned}
\vartheta_1'(0) &= \vartheta_2(0)\vartheta_3(0)\vartheta_4(0)\\
&= 2Gq^{\frac{1}{4}}\prod_{n=1}^{\infty}(1+2q^{2n}+q^{4n})\cdot G\prod_{n=1}^{\infty}(1-2q^{2n-1}+q^{4n-2})\\
&\qquad\qquad\qquad \cdot G\prod_{n=1}^{\infty}(1+2q^{2n-1}+q^{4n-2})\\
&= 2G^3q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1+q^{2n})^2(1-q^{2n-1})^2(1+q^{2n-1})^2\\
&= 2q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1-q^{2n})^3(1+q^{2n})^2(1-q^{2n-1})^2(1+q^{2n-1})^2\\
&= 2q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{4n})^2(1-q^{4n-2})^2.
\end{aligned}$$

Now, by using expressions (3.1) and (3.2), we have

$$\begin{aligned}
\frac{\sigma(z+\varepsilon)}{\sigma(z)\sigma(\varepsilon)}\exp(-z\zeta(\varepsilon)) &= \frac{2\omega}{\pi}\exp\left(\frac{\eta(z+\varepsilon)^2}{2\omega}\right)\frac{\vartheta_1\left(\frac{z+\varepsilon}{2\pi}\right)}{\vartheta_1'(0)}\\
&\quad\times\frac{\pi^2}{4\omega^2}\exp\left(-\frac{\eta z^2}{2\omega}\right)\frac{\vartheta_1'(0)}{\vartheta_1\left(\frac{\pi z}{2\omega}\right)}\\
&\quad\times\exp\left(-\frac{\eta\varepsilon^2}{2\omega}\right)\frac{\vartheta_1'(0)}{\vartheta_1\left(\frac{\pi\varepsilon}{2\omega}\right)}\exp\left(-z\left(\frac{\eta\varepsilon}{\omega}+\frac{\pi}{2\omega}\frac{\vartheta_1'(\frac{\pi\varepsilon}{2\omega})}{\vartheta_1(\frac{\pi\varepsilon}{2\omega})}\right)\right)\\
&= \frac{\pi}{2\omega}\exp\left(\frac{2\eta z\varepsilon}{2\omega}\right)\frac{\vartheta_1\left(\frac{\pi(z+\varepsilon)}{2\omega}\right)\vartheta_1'(0)}{\vartheta_1\left(\frac{\pi z}{2\omega}\right)\vartheta_1\left(\frac{\pi\varepsilon}{2\omega}\right)}\exp\left(-\frac{z\eta\varepsilon}{\omega}-\frac{\pi z}{2\omega}\frac{\vartheta_1'\left(\frac{\pi\varepsilon}{2\omega}\right)}{\vartheta_1\left(\frac{\pi\varepsilon}{2\omega}\right)}\right)\\
&= \frac{\pi}{2\omega}\frac{\vartheta_1\left(\frac{\pi(z+\varepsilon)}{2\omega}\right)\vartheta_1'(0)}{\vartheta_1\left(\frac{\pi z}{2\omega}\right)\vartheta_1\left(\frac{\pi\varepsilon}{2\omega}\right)}\exp\left(-\frac{\pi z}{2\omega}\frac{\vartheta_1'(\frac{\pi\varepsilon}{2\omega})}{\vartheta_1(\frac{\pi\varepsilon}{2\omega})}\right)\\
&= \frac{\pi}{2\omega}\frac{\vartheta_1\left(\nu+\frac{\pi\varepsilon}{2\omega}\right)\vartheta_1'(0)}{\vartheta_1\left(\frac{\pi z}{2\omega}\right)\vartheta_1\left(\frac{\pi\varepsilon}{2\omega}\right)}\exp\left(-\nu\frac{\vartheta_1'(\frac{\pi\varepsilon}{2\omega})}{\vartheta_1(\frac{\pi\varepsilon}{2\omega})}\right). \qquad (3.3)
\end{aligned}$$

Following Whittaker and Watson([8], §21.62, pp. 479-480) we write

$$\vartheta_1(u\vartheta_3^{-2}(0)|\tau) = -iq^{-\frac{1}{4}}e^{i\frac{\pi u}{2\mathbf{K}}}\Theta(u+i\mathbf{K}) = H(u). \qquad (3.4)$$

Now by replacing $u\vartheta_3^{-2}$ by $(u+\alpha)\vartheta_3^{-2}(0) \equiv \nu + \frac{\pi\varepsilon}{2\omega}$ in equation (3.4), we get

$$\vartheta_1(\nu+\frac{\pi\varepsilon}{2\omega}|\tau) = \vartheta_1((u+\alpha)\vartheta_3^{-2}(0)|\tau) = H(u+\alpha), \qquad (3.5)$$

where, $\vartheta_3(0) = \left(\frac{2\mathbf{K}}{\pi}\right)^{\frac{1}{2}} = 1 + 2q + 2q^4 + 2q^9 + \ldots$ (Whittaker and Watson [8], §21.61, p.479). Thus, we see that $\vartheta_3^{-2}(0) = \frac{\pi}{2\mathbf{K}}, \nu = \frac{\pi u}{2\mathbf{K}}, \frac{\pi\alpha}{2\mathbf{K}} = \frac{\pi\varepsilon}{2\omega}$ and $\alpha = \frac{\varepsilon\mathbf{K}}{\omega}$. Therefore, we can now re-write equation (3.3) using (3.5) as

$$\begin{aligned}
\frac{\sigma(z+\varepsilon)}{\sigma(z)\sigma(\varepsilon)}\exp(-z\zeta(\varepsilon)) &= \frac{\vartheta_3^{-2}(0)\frac{\alpha}{\varepsilon}\vartheta_1'(0)H(u+\alpha)\exp\left(-u\frac{\pi}{2\mathbf{K}}\frac{\vartheta_1'\left(\frac{\pi\alpha}{2\mathbf{K}}\right)}{\vartheta_1\left(\frac{\pi\alpha}{2\mathbf{K}}\right)}\right)}{\vartheta_1(u\vartheta_3^{-2}(0)|\tau)\vartheta_1(\alpha\vartheta_3^{-2}(0)|\tau)} \\
&= \frac{\alpha}{\varepsilon}\cdot\frac{\vartheta_3^{-2}(0)\vartheta_1'(0)H(u+\alpha)\exp\left(-u\vartheta_3^{-2}(0)\frac{\vartheta_1'\left(\alpha\vartheta_3^{-2}(0)|\tau\right)}{\vartheta_1'\left(\alpha\vartheta_3^{-2}(0)|\tau\right)}\right)}{\vartheta_1(u\vartheta_3^{-2}(0)|\tau)\vartheta_1(\alpha\vartheta_3^{-2}(0)|\tau)} \\
&= \frac{\alpha\vartheta_1'(0)\vartheta_3^{-2}(0)}{\varepsilon}\frac{H(u+\alpha)\exp\left(-uZ(\alpha)\right)\exp\left(-u\frac{\mathrm{d}}{\mathrm{d}\alpha}(\ln sn(\alpha,k)\right)}{\vartheta_1(u\vartheta_3^{-2}(0)|\tau)\vartheta_1(\alpha\vartheta_3^{-2}(0)|\tau)} \\
&= Af(u,\alpha)\frac{H(u+\alpha)}{\Theta(u)\Theta(\alpha)}\exp\left(-uZ(\alpha)\right), \qquad (3.6)
\end{aligned}$$

where $A = \frac{\alpha\vartheta_1'(0)\vartheta_3^{-2}(0)}{k\varepsilon}$ and

$$\begin{aligned}
f(u,\alpha) &= ns(u,k)ns(\alpha,k)\exp\left(-u\frac{\mathrm{d}}{\mathrm{d}\alpha}(\ln sn(\alpha,k))\right) \\
&= ns(u,k)ns(\alpha,k)\exp\left(-u\,cs(\alpha,k)\right)
\end{aligned}$$

because by (2.11) and (3.4)

$$\vartheta_1(\alpha\vartheta_3^{-2}(0)|\tau) = H(\alpha) = \sqrt{k}\Theta(\alpha)sn(\alpha,k).$$

By replacing ε with $-\varepsilon$ in equation (3.6) the complementary solution of equation (1.1) is obtained as

$$\frac{\sigma(z-\varepsilon)}{\sigma(z)\sigma(\varepsilon)}\exp(z\zeta(\varepsilon)) = Af(u,\alpha)\frac{H(u-\alpha)}{\Theta(u)\Theta(\alpha)}\exp\left(uZ(\alpha)\right). \qquad (3.7)$$

Hence, combining the two linearly independent solutions (3.6) and (3.7) we have

$$\frac{\sigma(z\pm\varepsilon)}{\sigma(z)\sigma(\varepsilon)}\exp(\mp z\zeta(\varepsilon)) = Af(u,\alpha)\frac{H(u\pm\alpha)}{\Theta(u)\Theta(\alpha)}\exp\left(\mp uZ(\alpha)\right).$$

Hence the result. □

4 Conclusion

In this paper, the theory of elliptic functions has been adapted in the study of the relationship between solutions of Lamé equations in the Weierstrass and Jacobi forms. The solutions have been found to be equal up to a multiplicative elliptic function.

References

[1] M. Abramowitz and I.A. Stegun (eds.), *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables,* National Bureau of Standards (Applied Mathematics Series 55), Washington D.C., 1972.

[2] N.I. Akhiezer, *Elements of the Theory of Elliptic Functions,* American Mathematical Society, Translations of Mathematical Monographs, Vol. 79, 1990. https://doi.org/10.1090/mmono/079

[3] A.L. Baker, *Elliptic Functions,* John Wiley and Sons, New York, 1890.

[4] U.N. Bassey and U.S. Idiong, Lamé Equation in the Weierstrass form and Exact Solvability, *International Journal of Differential Equations and Applications*, **15** (2016), no. 2, 77-89.

[5] P.F. Byrd and M.D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists,* Springer-Verlag, New York, Heidelberg, Berlin, 1972.

[6] A. Erdélyi(Ed.), W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher Transcendental Functions*, (Bateman Manuscript Project) Vol III, McGraw-Hill, 1955.

[7] E.L. Ince, *Ordinary Differential Equations*, Dover, New York, 1956.

[8] E.T. Whittaker and G.N. Watson, *A Course in Modern Analysis,* Cambridge University Press, New York, 1902.

Received: March 3, 2018; Published: May 17, 2018