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Almost Periodic Solutions for Impulsive Lasota-Wazewska Model with Discontinuous Coefficients

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Abstract

As we know, the coefficients of impulsive Lasota-Wazewska model are always assumed to be continuous, say, periodic and almost periodic. In this paper we consider the case when the coefficients are piecewise almost periodic, and establish an existence and uniqueness theorem of piecewise almost periodic positive solution for this model.

Mathematics Subject Classifications: 34C27, 34A37

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1 Introduction

In 1976, Wazewska-Czyzewska and Lasota [1] introduced the following reduced Lasota-Wazewska model:

$$x'(t) = -\alpha x(t) + \beta e^{-\gamma x(t-h)}, \tag{1.1}$$

which describes the dynamics of red blood cells production, where $x(t)$ denotes the number of red blood cells in the circulation, $\alpha > 0$ is average part of

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red blood cells population being destroyed in the time unit, $\beta > 0$ is the constant connected with demand for oxygen, $\gamma > 0$ characterizes excitability of haematopoietic system, and h is the time delay of the haematopoietic system. Then being regarded as an important general biological system, this model and its various generalized forms have been extensively studied by lots of author(see e.g. [2–5]).

The theory of impulsive differential equation has been well developed in recent years (see [6–10]). If the impulsive factors of the environment are incorporated into the biological dynamic models, the models must be governed by impulsive differential equations. As a result, lots of works are devoted to the study of the following impulsive Lasota-Wazewska model with multiple time-varying delays (see e.g. [11–13]):

$$\begin{cases} x'(t) = -\alpha(t)x(t) + \displaystyle\sum_{j=1}^{m} \beta_j(t)e^{-\gamma_j(t)x(t-\eta_j(t))}, t \neq t_k, \\ \Delta x(t_k) = x(t_k+0) - x(t_k-0) = \alpha_k x(t_k) + \nu_k. \end{cases} \tag{1.2}$$

We notice that the coefficients $\alpha, \beta_j, \gamma_j, \eta_j, j = 1, 2, \cdots, m$ are always assumed to be continuous in the literature. However, we have to face the case when the coefficients are discontinuous. This is natural in the real life. For instance, in the dynamics of red blood cells production, the red blood cells population being destroyed, the demand for oxygen or the excitability of haematopoietic system may also be changed instantaneously because of the instantaneously changes of the environment. As a result, the model (1.2) with discontinuous coefficients is an interesting and important topic to study. This is the main motivation for this work. We consider system (1.2) in the case when the coefficients are piecewise almost periodic, and establish an existence and uniqueness theorem of piecewise almost periodic positive solution for this model.

2 Preliminaries

A positive solution $x(t)$ of (1.2) means that $x(t)$ satisfies (1.2) and $x(t) > 0, t \in \mathbb{R}$. Throughout this paper, we denote by $\mathbb{T}$ the set of real sequences $T = \{t_i\}_{i\in\mathbb{Z}}$ such that $\sigma = \inf_{i\in\mathbb{Z}}(t_{i+1} - t_i) > 0$ and $\kappa = \sup_{i\in\mathbb{Z}}(t_{i+1} - t_i) < \infty$. For $T \in \mathbb{T}$, let $PC_T(\mathbb{R})$ be the space of bounded piecewise continuous functions $\phi : \mathbb{R} \to \mathbb{R}$ such that ϕ is continuous at t for $t \notin \{t_i\}$ and $\phi(t_i) = \phi(t_i^-), i \in \mathbb{Z}$. For convenience, we set $\phi^- = \inf_{t\in\mathbb{R}} \phi(t)$.

Definition 2.1 ([14]) A function $\phi \in PC_T(\mathbb{R})$ is said to be piecewise almost periodic if the following conditions are fulfilled:

(i) $\{t_i^j = t_{i+j} - t_i\}_{i\in\mathbb{Z}}$, $j = 0, \pm1, \pm2, \cdots$ are equipotentially almost periodic, i.e. for any $\varepsilon > 0$, there exists a relatively dense set of ε-periods, that are common to all the sequences $\{t_i^j\}_{i\in\mathbb{Z}}$, $j = 0, \pm1, \pm2, \cdots$.

(ii) For $\varepsilon > 0$, there exists a positive number $\delta = \delta(\varepsilon)$ such that if $t', t'' \in (t_{i-1}, t_i]$ for some $i \in \mathbb{Z}$ and $|t' - t''| < \delta$, then $|\phi(t') - \phi(t'')| < \varepsilon$.

(iii) For $\varepsilon > 0$, there exists a relatively dense set $Q_\varepsilon \subset \mathbb{R}$ such that if $\tau \in Q_\varepsilon$, then $|\phi(t+\tau) - \phi(t)| < \varepsilon$ for all $t \in \mathbb{R}$, $|t - t_i| > \varepsilon, i \in \mathbb{Z}$. The number τ is called ε-period of ϕ.

Denote by $AP_T(\mathbb{R})$ the space of piecewise almost periodic functions, which is a Banach space endowed with the sup norm $\|\cdot\|$. We always assume that $\alpha, \beta_j, \gamma_j \in AP_T(\mathbb{R}), j = 1, 2, \cdots, m$ are nonnegative with $\alpha^- = \inf_{t\in\mathbb{R}} \alpha(t) > 0$, and $\{\alpha_k\}_{k\in\mathbb{Z}}, \{\nu_k\}_{k\in\mathbb{Z}}$ are almost periodic with $-1 < \alpha_k < 0, k \in \mathbb{Z}$. We denote $\bar{\alpha} = \inf_{k\in\mathbb{Z}} \alpha_k$ and $\|\nu\| = \sup_{k\in\mathbb{Z}} |\nu_k|$.

Together with (1.2), we consider the linear system

$$\begin{cases} x'(t) = -\alpha(t)x(t), & t \in \mathbb{R}, t \neq t_k, \\ \Delta x(t_k) = \alpha_k x(t_k), & k \in \mathbb{Z}. \end{cases} \tag{2.1}$$

Similar as that in [6], we can get the Cauchy matrix (actually one order) of (2.1):

$$W(t,s) = \begin{cases} e^{-\int_s^t \alpha(\theta)d\theta}, & t_{k-1} < s \leq t \leq t_k, \\ \displaystyle\prod_{j=m}^{k} (1+\alpha_j) e^{-\int_s^t \alpha(\theta)d\theta}, & t_{m-1} < s \leq t_m \leq t_k < t \leq t_{k+1}, \end{cases}$$

and the solution of system (2.1) are in the form $x(t; t_0, x_0) = W(t, t_0)x_0$ for $t_0, x_0 \in \mathbb{R}$.

The following lemma can be obtained by a slight modification of the proof of [11, Lemma 1.7], and we omit the details.

Lemma 2.1 *Let $u \in AP_T(\mathbb{R})$. For $\varepsilon > 0$, there exist $\varepsilon_1 \in (0, \varepsilon)$, relatively dense sets $\Omega \subset \mathbb{R}$ and $Q \subset \mathbb{Z}$ such that for $\tau \in \Omega$, $q \in Q$, $t \in \mathbb{R}$ with $|t - t_i| > \varepsilon, i \in \mathbb{Z}$ and $k \in \mathbb{Z}$, the following relations are fulfilled:*

(a) $|\rho(t+\tau) - \rho(t)| < \varepsilon$, *where* $\rho = \alpha, u, \beta_j, \gamma_j, j = 1, 2, \ldots, m$.

(b) $|\alpha_{k+q} - \alpha_k| < \varepsilon, |\nu_{k+q} - \nu_k| < \varepsilon$.

(c) $|t_k^q - \tau| < \varepsilon_1$.

Lemma 2.2 *For $t, s \in \mathbb{R}$ with $t \geq s$, we have $0 < W(t,s) \leq e^{-\alpha^-(t-s)}$. Moreover, for $\varepsilon > 0$, let Ω be as in Lemma 2.1, there exists a constant $\lambda > 0$ such that for $\tau \in \Omega$, $|t - t_k| > \varepsilon, |s - t_k| > \varepsilon, k \in \mathbb{Z}$,*

$$|W(t+\tau, s+\tau) - W(t,s)| \leq \lambda \varepsilon e^{-\alpha^-(t-s)/2}. \tag{2.2}$$

Proof. By a direct calculation we can get that $0 < W(t,s) \leq e^{-\alpha^-(t-s)}$ for $t, s \in \mathbb{R}$ with $t \geq s$. To prove the rest of this lemma, we use the notations in Lemma 2.1. Let $\tau \in \Omega, q \in Q$ and $|t - t_k| > \varepsilon$, $|s - t_k| > \varepsilon$, $k \in \mathbb{Z}$. It follows from Lemma 2.1 that

$$|t + \tau - t_{k+q}| > \varepsilon - \varepsilon_1 > 0 \quad \text{and} \quad |s + \tau - t_{k+q}| > \varepsilon - \varepsilon_1 > 0, \quad k \in \mathbb{Z}. \tag{2.3}$$

Let N be the smallest integer bigger than $(t-s)/\sigma$. Then there are at most N points in $[s,t] \cap T$ and

$$N \leq 1 + \frac{t-s}{\sigma} \leq 1 + \frac{2}{\sigma\alpha^-} e^{\alpha^-(t-s)/2} \leq \left(1 + \frac{2}{\sigma\alpha^-}\right) e^{\alpha^-(t-s)/2} \triangleq M_0 e^{\alpha^-(t-s)/2}. \tag{2.4}$$

Now we have two cases to be considered: $[s,t] \cap T = \emptyset$ and $[s,t] \cap T = \{r_1, r_2, \cdots, t_p\}$ for some integer $p \leq N$.

Case 1. Assume that $[s,t] \cap T = \emptyset$. By (2.3) and Lemma 2.1, we can get that $[s+\tau, t+\tau] \cap T = \emptyset$ and

$$|\alpha(\theta + \tau) - \alpha(\theta)| < \varepsilon \quad \text{for } \theta \in [s,t]. \tag{2.5}$$

It is easy to verify that the following inequality holds:

$$|1 - e^{\pm ab}| \leq e^a b \quad \text{for } a > 0, 0 < b \leq 1. \tag{2.6}$$

We may assume that $\varepsilon < \min\{\alpha^-/2, \alpha^-/(4\|\alpha\|/\sigma + 2)\}$. Then by the definition of W, (2.5) and (2.6),

$$\begin{aligned}
|W(t+\tau, s+\tau) - W(t,s)| &= |e^{-\int_{s+\tau}^{t+\tau} \alpha(\theta)d\theta} - e^{-\int_s^t \alpha(\theta)d\theta}| \\
&= |1 - e^{-\int_s^t (\alpha(\theta) - \alpha(\theta+\tau))d\theta}| e^{-\int_s^t \alpha(\theta)d\theta} \\
&\leq |1 - e^{\pm\varepsilon(t-s)}| e^{-\alpha^-(t-s)} \\
&\leq (2/\alpha^-)\varepsilon e^{\alpha^-(t-s)/2} e^{-\alpha^-(t-s)} \\
&= (2/\alpha^-)\varepsilon e^{-\alpha^-(t-s)/2}.
\end{aligned}$$

That is (2.2) holds with $\lambda = 2/\alpha^-$.

Case 2. Assume that $[s,t] \cap T = \{r_1, r_2, \cdots, r_p\}$ for some integer $p \leq N$. By (2.3) and Lemma 2.1, we have $[s+\tau, t+\tau] \cap T = \{r_{1+q}, r_{2+q}, \cdots, r_{p+q}\}$ and

$$|\alpha_{j+q} - \alpha_j| < \varepsilon, \quad j = 1, \cdots, p. \tag{2.7}$$

Set $U_{ts} = \bigcup_{k=1}^{p}[r_k - \varepsilon, r_k + \varepsilon]$. Then the measure $mU_{ts} \leq 2N\varepsilon \leq 2(1 + (t - s)/\sigma)\varepsilon$. By Lemma 2.1,

$$|\alpha(\theta + \tau) - \alpha(\theta)| < \varepsilon \quad \text{for } \theta \in [s, t] \setminus U_{ts}.$$

Thus we have

$$\begin{aligned}\left|\int_s^t (\alpha(\theta) - \alpha(\theta + \tau))d\theta\right| &\leq \left(\int_{[s,t]\cap U_{ts}} + \int_{[s,t]\setminus U_{ts}}\right) |\alpha(\theta) - \alpha(\theta + \tau)|d\theta \\ &\leq \|\alpha\| mU_{ts} + (t - s)\varepsilon \\ &\leq 2\|\alpha\|\varepsilon + (2\|\alpha\|/\sigma + 1)(t - s)\varepsilon.\end{aligned}$$

Then by (2.6),

$$\begin{aligned}|e^{-\int_{s+\tau}^{t+\tau} \alpha(\theta)d\theta} - e^{-\int_s^t \alpha(\theta)d\theta}| &= |e^{-\int_s^t (\alpha(\theta+\tau)-\alpha(\theta))d\theta} - 1|e^{-\int_s^t \alpha(\theta)d\theta} \\ &\leq |e^{\pm(2\|\alpha\|\varepsilon+(2\|\alpha\|/\sigma+1)(t-s)\varepsilon)} - 1|e^{-\alpha^-(t-s)} \\ &\leq \frac{4\|\alpha\|/\sigma + 2}{\alpha^-}\varepsilon e^{\|\alpha\|\alpha^-/(2\|\alpha\|/\sigma+1)+\alpha^-(t-s)/2}e^{-\alpha^-(t-s)} \\ &= \frac{4\|\alpha\|/\sigma + 2}{\alpha^-}e^{\|\alpha\|\alpha^-/(2\|\alpha\|/\sigma+1)}\varepsilon e^{-\alpha^-(t-s)/2} \\ &\triangleq M_1\varepsilon e^{-\alpha^-(t-s)/2}. \qquad (2.8)\end{aligned}$$

For the convenience, denote $\Lambda = \prod_{j=1}^{p}(1 + \alpha_j)$ and $\Lambda_q = \prod_{j=1}^{p}(1 + \alpha_{j+q})$. Noticing that $(1 + \alpha_k) \in (0, 1)$ for $k \in \mathbb{Z}$, then by (2.4) and (2.7),

$$|\Lambda_q - \Lambda| \leq \sum_{j=k}^{N} |\alpha_{j+q} - \alpha_j| < N\varepsilon \leq M_0\varepsilon e^{\alpha^-(t-s)/2}. \qquad (2.9)$$

Thus by the definition of W, (2.8) and (2.9),

$$\begin{aligned}&|W(t + \tau, s + \tau) - W(t, s)| = \left|\Lambda_q e^{-\int_{s+\tau}^{t+\tau} \alpha(\theta)d\theta} - \Lambda e^{-\int_s^t \alpha(\theta)d\theta}\right| \\ &\leq |\Lambda_q - \Lambda|e^{-\int_s^t \alpha(\theta)d\theta} + \left|e^{-\int_{s+\tau}^{t+\tau} \alpha(\theta)d\theta} - e^{-\int_s^t \alpha(\theta)d\theta}\right| \\ &\leq M_0\varepsilon e^{\alpha^-(t-s)/2}e^{-\alpha^-(t-s)} + M_1\varepsilon e^{-\alpha^-(t-s)/2} \\ &= (M_0 + M_1)\varepsilon e^{-\alpha^-(t-s)/2}.\end{aligned}$$

Then (2.2) holds with $\lambda = M_0 + M_1$.

3 Existence of piecewise almost periodic solutions

Lemma 3.1 *Let $u \in AP_T(\mathbb{R})$ with $u(t) \geq 0, t \in \mathbb{R}$ and*

$$F(t) = \int_{-\infty}^{t} W(t, s)\beta_j(s)e^{-\gamma_j(s)u(s+h)}ds$$

for some $j \in \{1, \cdots, m\}$*. Then* $F \in AP_T(\mathbb{R})$*.*

Proof: For the convenient of whiting, we denote $l_j(s) = e^{-\gamma_j(s)u(s+h)}, s \in \mathbb{R}, j = 1, 2, \cdots, m$. Then $0 < l_j(s) \le 1$. It is easy to see that F is uniformly continuous.

For $\varepsilon > 0$, there exists $\eta > \sigma$ such that $e^{-\eta\alpha^-}/\alpha^- < \varepsilon$. Let N be the smallest integer bigger than η/σ. Then for $t \in \mathbb{R}$, we may assume that $T \cap [t-\eta, t] = \{r_1, r_2, \cdots, r_p\}$, $(T-h) \cap [t-\eta, t] = \{r'_1, r'_2, \cdots, r'_{p'}\}$ with $p, p' \le N$. Here $T - h = \{t_k - h\}_{k\in\mathbb{Z}}$. Set

$$A_t = \bigcup_{k=1}^{p} \left[r_k - \frac{\varepsilon}{4N}, r_k + \frac{\varepsilon}{4N}\right] \bigcup_{k=1}^{p'} \left[r'_k - \frac{\varepsilon}{4N}, r'_k + \frac{\varepsilon}{4N}\right].$$

Obviously, the measure $mA_t \le \varepsilon$. By Lemma 2.1 and 2.2, for $\varepsilon/(4N)$, there exists a relatively dense set $\Omega \subset \mathbb{R}$ such that for $\tau \in \Omega$,

$$\begin{aligned}
&|\rho(s+\tau) - \rho(s)| < \varepsilon/(4N), \quad \text{where } \rho = \alpha, u, \beta_j, \gamma_j, j = 1, 2, \ldots, m,\\
&|W(t+\tau, s+\tau) - W(t,s)| \le \lambda\varepsilon/(4N)e^{-\alpha^-(t-s)/2} \le \lambda\varepsilon/(4N),
\end{aligned}$$

for $t \in \mathbb{R}$ with $|t - t_i| > \varepsilon/(4N)$ and $|s - t_i| > \varepsilon/(4N), i \in \mathbb{Z}$. Noticing that $W(t,s), l_j(s) \in (0,1]$, then for $\tau \in \Omega$, $|t - t_i| > \varepsilon > \varepsilon/(4N), i \in \mathbb{Z}$, $s \in [t-\eta, t] \setminus A_t$,

$$\begin{aligned}
&|W(t+\tau, s+\tau)\beta_j(s+\tau)l_j(s+\tau) - W(t,s)\beta_j(s)l_j(s)|\\
&\le \|\beta_j\||W(t+\tau, s+\tau) - W(t,s)| + |\beta_j(s+\tau) - \beta_j(s)| + \|\beta_j\||l_j(s+\tau) - l_j(s)|\\
&\le \|\beta_j\|\lambda\varepsilon/(4N) + \varepsilon/(4N) + \|\beta_j\||\gamma_j(s+\tau)u(s+h+\tau) - \gamma_j(s)u(s+h)|\\
&\le (\|\beta_j\|\lambda + 1)\varepsilon/(4N) + \|\beta_j\||(\|u\||\gamma_j(s+\tau) - \gamma_j(s)| + \|\gamma_j\||u(s+h+\tau) - u(s+h)|)\\
&\le (\|\beta_j\|(\lambda + \|u\| + \|\gamma_j\|) + 1)\varepsilon/(4N),
\end{aligned}$$

and hence

$$\begin{aligned}
&\int_{t-\eta}^{t} |W(t+\tau, s+\tau)\beta_j(s+\tau)l_j(s+\tau) - W(t,s)\beta_j(s)l_j(s)|ds\\
&\le \int_{[t,t-\eta]\cap A_t} \|\beta_j\|ds + \int_{[t,t-\eta]\setminus A_t} (\|\beta_j\|(\lambda + \|u\| + \|\gamma_j\|) + 1)\varepsilon/(4N)ds\\
&\le \|\beta_j\|\varepsilon + (\|\beta_j\|(\lambda + \|u\| + \|\gamma_j\|) + 1)\eta\varepsilon/(4N)\\
&\le [\|\beta_j\| + (\|\beta_j\|(\lambda + \|u\| + \|\gamma_j\|) + 1)\sigma/4]\varepsilon \triangleq M_2\varepsilon.
\end{aligned}$$

Therefore, by Lemma 2.2, for $\tau \in \Omega$, $|t - t_i| > \varepsilon, i \in \mathbb{Z}$,

$$\begin{aligned}
|F(t+\tau) - F(t)| &= \left| \int_{-\infty}^{t+\tau} W(t+\tau, s)\beta_j(s)l_j(s)ds - \int_{-\infty}^{t} W(t,s)\beta_j(s)l_j(s)ds \right| \\
&\leq \int_{-\infty}^{t} |W(t+\tau, s+\tau)\beta_j(s+\tau)l_j(s+\tau) - W(t,s)\beta_j(s)l_j(s)|ds \\
&\leq \int_{-\infty}^{t-\eta} \|\beta_j\| e^{-\alpha^-(t-s)}ds + M_2\varepsilon \\
&= \|\beta_j\| \frac{e^{-\eta\alpha^-}}{\alpha^-} + M_0\varepsilon \leq (\|\beta_j\| + M_2)\varepsilon.
\end{aligned}$$

This implies that $F \in AP_T(\mathbb{R})$.

For convenience, we denote

$$\begin{aligned}
K &= \frac{\sum_{j=1}^{m} \|\beta_j\|}{\alpha^-} + \frac{\|\nu\|}{1 - e^{-\sigma\alpha^-}}, \\
\underline{K} &= \frac{(1 - e^{-\|\alpha\|\sigma})(1 - e^{-\alpha^-\sigma})}{\|\alpha\|(1 - (1+\bar{\alpha})e^{-\|\alpha\|\kappa})} \sum_{j=1}^{m} \beta_j^- e^{-\|\gamma_j\|K} - \|\nu_k\|.
\end{aligned}$$

Theorem 3.1 *Assume that* $\underline{K} > 0$ *and* $\sum_{j=1}^{m} \|\beta_j\| < \alpha^-$. *Then equation (1.2) has a unique positive solution* $u \in AP_T(\mathbb{R})$ *such that* $0 < u(t) \leq K, t \in \mathbb{R}$.

Proof: Let $\mathcal{B} = \{u \in AP_T(\mathbb{R}) : 0 \leq u(t) \leq K, t \in \mathbb{R}\}$. For $u \in \mathcal{B}$, define

$$\Gamma u(t) = \int_{-\infty}^{t} W(t,s) \sum_{j=1}^{m} \beta_j(s) e^{-\gamma_j(s)u(s+h)}ds + \sum_{t_k<t} W(t, \tau_k)\nu_k = \Gamma_1 u(t) + \Gamma_2 u(t),$$

where $\Gamma_1 u(t)$ and $\Gamma_2 u(t)$ denote the integral term and the sum term, respectively. Then the proof is completed if Γ has a unique fixed point in $\mathcal{B}$.

By Lemma 2.2 and the definition of W, we can get that

$$|\Gamma u(t)| \leq \int_{-\infty}^{t} e^{-\alpha^-(t-s)} \sum_{j=1}^{m} \|\beta_j\| ds + \sum_{t_k<t} e^{-\alpha^-(t-t_k)}\nu_k \leq \frac{\sum_{j=1}^{m} \|\beta_j\|}{\alpha^-} + \frac{\|\nu\|}{1 - e^{-\sigma\alpha^-}} = K.$$

Assume that $t \in (t_n, t_{n+1}]$ for some $n \in \mathbb{Z}$, then by the definition of W,

$$\begin{aligned}
\int_{-\infty}^{t} W(t,s)ds &= \left(\int_{t_n}^{t} + \sum_{k=-\infty}^{n} \int_{t_{k-1}}^{t_k} \right) W(t,s)ds \\
&\geq \int_{t_n}^{t} e^{-\|\alpha\|(t-s)} ds + \sum_{k=-\infty}^{n} \int_{t_{k-1}}^{t_k} \prod_{j=k}^{n} (1+\alpha_j) e^{-\|\alpha\|(t-s)} ds \\
&= \frac{1-e^{-\|\alpha\|(t-t_n)}}{\|\alpha\|} + \sum_{k=-\infty}^{n} \prod_{j=k}^{n} (1+\alpha_j) \frac{e^{-\|\alpha\|(t-t_k)}(1-e^{-\|\alpha\|(t_k - t_{k-1})})}{\|\alpha\|} \\
&\geq \frac{1-e^{-\|\alpha\|\sigma}}{\|\alpha\|} \left(1 + \sum_{k=-\infty}^{n} (1+\bar{\alpha})^{n-k+1} e^{-\|\alpha\|\kappa(n-k+1)} \right) \\
&= \frac{1-e^{-\|\alpha\|\sigma}}{\|\alpha\|(1-(1+\bar{\alpha})e^{-\|\alpha\|\kappa})}.
\end{aligned}$$

Noticing that $\underline{K} > 0$, we have

$$\begin{aligned}
\Gamma u(t) &\geq \sum_{j=1}^{m} \beta_j^- e^{-\|\gamma_j\|K} \int_{-\infty}^{t} W(t,s)ds - \sum_{t_k<t} e^{-\alpha^-(t-t_k)} \|\nu\| \\
&\geq \frac{1-e^{-\|\alpha\|\sigma}}{\|\alpha\|(1-(1+\bar{\alpha})e^{-\|\alpha\|\kappa})} \sum_{j=1}^{m} \beta_j^- e^{-\|\gamma_j\|K} - \frac{\|\nu\|}{1-e^{-\sigma\alpha^-}} > 0.
\end{aligned}$$

Meanwhile, by Lemma 3.1, we get that $\Gamma_1 u \in AP_T(\mathbb{R})$. To prove that $\Gamma u \in \mathcal{B}$, we need only to prove that $\Gamma_2 u \in AP_T(\mathbb{R})$.

In fact, by Lemma 2.2 and the definition of W, it is easy to see that the sum in Γ_2 is convergent uniformly in $t \in \mathbb{R}$, and $\Gamma_2 u$ satisfies the condition (ii) of Definition 2.1. Next we use the notations in Lemma 2.1. For $\varepsilon > 0$, let $\tau \in \Omega$, $q \in Q$. For $t \in \mathbb{R}$ with $|t - t_i| > \varepsilon$, and $t_k \in T$ with $t_k < t$. Assume that $[t_k, t] \cap T = \{t_k, t_{k+1}, \cdots, t_p\}$ for some integer $p \geq k$. Then $p - k \leq (t - t_k)/\sigma$. It is easy to get from Lemma 2.1 that $[t_{k+q}, t+\tau] \cap T = \{t_{k+q}, t_{k+q+1}, \cdots, t_{p+q}\}$ and

$$|\alpha(\theta+\tau)-\alpha(\theta)| < \varepsilon, |\alpha_{i+q}-\alpha_i| < \varepsilon, |\nu_{i+q}-\nu_i| < \varepsilon, |t_{k+q}-t_k-\tau| < \varepsilon_1 < \varepsilon \quad (3.1)$$

for $\theta \in \mathbb{R}, |\theta - t_j| > \varepsilon, i, j \in \mathbb{Z}$. Set $U_{tk} = \bigcup_{j=k}^{p} [t_j - \varepsilon, t_j + \varepsilon]$. Then the measure $mU_{tk} \leq 2(p-k+1)\varepsilon \leq 2\varepsilon + 2(t-t_k)\varepsilon/\sigma$, and we have

$$\begin{aligned}
\left| \int_{t_k}^{t} (\alpha(\theta+\tau) - \alpha(\theta)) d\theta \right| &\leq \left(\int_{[t_k,t]\cap U_{tk}} + \int_{[t_k,t]\setminus U_{tk}} \right) |\alpha(\theta+\tau) - \alpha(\theta)| d\theta \\
&\leq \|\alpha\|(2\varepsilon + 2(t-t_k)\varepsilon/\sigma) + (t-t_k)\varepsilon \\
&\triangleq 2\|\alpha\|\varepsilon + M_3(t-t_k)\varepsilon.
\end{aligned}$$

Thus by (3.1),

$$\begin{aligned}\left|\int_{t_{k+q}}^{t+\tau}\alpha(\theta)d\theta-\int_{t_k}^{t}\alpha(\theta)d\theta\right| &\le \int_{t_{k+q}}^{t_k+\tau}\alpha(\theta)d\theta+\int_{t_k}^{t}|\alpha(\theta+\tau)-\alpha(\theta)|d\theta\\ &\le \|\alpha\|\varepsilon+2\|\alpha\|\varepsilon+M_3(t-t_k)\varepsilon\\ &= 3\|\alpha\|\varepsilon+M_3(t-t_k)\varepsilon.\end{aligned}$$

So by (2.6) (we may assume that $\varepsilon<\alpha^-/(2M_3)$),

$$\begin{aligned}\left|e^{-\int_{t_{k+q}}^{t+\tau}\alpha(\theta)d\theta}-e^{-\int_{t_k}^{t}\alpha(\theta)d\theta}\right| &= \left|e^{-\left(\int_{t_{k+q}}^{t+\tau}\alpha(\theta)d\theta-\int_{t_k}^{t}\alpha(\theta)d\theta\right)}-1\right|e^{-\int_{t_k}^{t}\alpha(\theta)d\theta}\\ &\le \left|e^{\pm(3\|\alpha\|+M_3(t-t_k))\varepsilon}-1\right|e^{-\alpha^-(t-t_k)}\\ &\le \frac{2M_3}{\alpha^-}\varepsilon e^{3\|\alpha\|\alpha^-/(2M_3)+\alpha^-(t-t_k)/2}e^{-\alpha^-(t-t_k)}\\ &\triangleq M_4\varepsilon e^{-\alpha^-(t-t_k)/2}.\end{aligned}$$

Denote $\Lambda'=\prod_{j=k}^{N}(1+\alpha_j)$ and $\Lambda'_q=\prod_{j=k}^{N}(1+\alpha_{j+q})$. As (2.9), we can get

$$|\Lambda'_q-\Lambda'|\le M_5\varepsilon e^{\alpha^-(t-t_k)/2}$$

for some constant $M_5>0$. Now by the definition of W and the fact that $|\Lambda_q|\le 1$,

$$\begin{aligned}|W(t+\tau,t_{k+q})-W(t,t_k)| &= \left|\Lambda_q e^{-\int_{t_{k+q}}^{t+\tau}\alpha(\theta)d\theta}-\Lambda e^{-\int_{t_k}^{t}\alpha(\theta)d\theta}\right|\\ &\le \left|e^{-\int_{t_{k+q}}^{t+\tau}\alpha(\theta)d\theta}-e^{-\int_{t_k}^{t}\alpha(\theta)d\theta}\right||\Lambda_q|+|\Lambda_q-\Lambda|e^{-\int_{t_k}^{t}\alpha(\theta)d\theta}\\ &\le M_4\varepsilon e^{-\alpha^-(t-t_k)/2}+M_5\varepsilon e^{\alpha^-(t-t_k)/2}e^{-\alpha^-(t-t_k)}\\ &= (M_4+M_5)\varepsilon e^{-\alpha^-(t-t_k)/2}.\end{aligned}$$

Then by (3.1) and Lemma 2.2,

$$\begin{aligned}|\Gamma_2(t+\tau)-\Gamma_2(t)| &= \left|\sum_{t_k<t}(W(t+\tau,t_{k+q})\nu_{k+q}-W(t,t_k)\nu_k)\right|\\ &\le \sum_{t_k<t}(|W(t+\tau,t_{k+q})-W(t,t_k)||\nu_{k+q}|+|W(t,t_k)||\nu_{k+q}-\nu_k|)\\ &\le \sum_{t_k<t}\left(\|\nu\|(M_4+M_5)\varepsilon e^{-\alpha^-(t-t_k)/2}+\varepsilon e^{-\alpha^-(t-t_k)}\right)\\ &\le (\|\nu\|(M_4+M_5)+1)\varepsilon\sum_{t_k<t}e^{-\alpha^-(t-t_k)/2}\\ &\le \frac{\|\nu\|(M_4+M_5)+1}{1-e^{-\sigma\alpha^-/2}}\varepsilon.\end{aligned}$$

This implies that $\Gamma_2 u \in AP_T(\mathbb{R})$, and then Γ is from $\mathcal{B}$ to $\mathcal{B}$.

Let $\phi, \varphi \in \mathcal{B}, t \in \mathbb{R}$, by Lemma 2.2

$$
\begin{aligned}
|\Gamma\phi(t) - \Gamma\varphi(t)| &\leq \int_{-\infty}^{t} W(t,s) \sum_{j=1}^{m} \beta_j(s) \left| e^{-\gamma_j(s)\phi(s+h)} - e^{-\gamma_j(s)\varphi(s+h)} \right| ds \\
&\leq \int_{-\infty}^{t} e^{-\alpha^-(t-s)} \sum_{j=1}^{m} \|\beta_j\| \|\phi - \varphi\| ds \leq \frac{\sum_{j=1}^{m} \|\beta_j\|}{\alpha^-} \|\phi - \varphi\|.
\end{aligned}
$$

Then it follows from the hypothesis of this theorem that $\Gamma : \mathcal{B} \to \mathcal{B}$ is contracting. So Γ has a unique fixed point $u \in \mathcal{B}$. This completes the proof.

Remark 3.1 (i) We note that model (1.2) with continuous almost periodic coefficients is investigated in [11], and similar result was obtained, but the solution considered there is not necessarily positive.

(ii) Denote

$$
\underline{K}' = \frac{(1 - e^{-\alpha^- \sigma})^2}{\|\alpha\|} \sum_{j=1}^{m} \beta_j^- e^{-\|\gamma_j\| K} - \|\nu_k\|.
$$

Then $\underline{K} \geq \underline{K}'$, and Theorem 3.1 holds if the condition $\underline{K} > 0$ is replaced by $\underline{K}' > 0$, which is stricter but simpler and easier to be verified.

We close this work by an example.

Example 3.1 Consider the following impulsive Lasota-Wazewska model:

$$
\begin{cases} x'(t) = -\alpha(t)x(t) + \beta_1(t)e^{-\gamma_1(t)x(t-h)} + \beta_2(t)e^{-\gamma_2(t)x(t-h)}, & t \neq \tau_k, \\ \Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k - 0) = \alpha_k x(\tau_k) + \nu_k, \end{cases} \tag{3.2}
$$

where $T = \{\tau_k\}$ with $\tau_k = k + |\sin k - \sin \sqrt{2}k|/4, k \in \mathbb{Z}$, $\{\alpha_k\}$ and $\{\nu_k\}$ are nonnegative almost periodic sequence with $-1 < \alpha_k < 0, k \in \mathbb{Z}$, $\|\nu\| = 1/4$ and

$$
\alpha(t) = \begin{cases} 10 + \cos^2 t, & t \in (\tau_{2n-1}, \tau_{2n}], n \in \mathbb{Z}, \\ 10 + \cos^2 2t, & t \in (\tau_{2n}, \tau_{2n+1}], n \in \mathbb{Z}, \end{cases}
$$

$$
\beta_1(t) = \begin{cases} 3 + |\sin \sqrt{2}t|, & t \in (\tau_{2n-1}, \tau_{2n}], n \in \mathbb{Z}, \\ 3 + |\sin \sqrt{3}t|, & t \in (\tau_{2n}, \tau_{2n+1}], n \in \mathbb{Z}, \end{cases}
$$

$$
\beta_2(t) = \begin{cases} 4 + |\sin \sqrt{5}t|, & t \in (\tau_{2n-1}, \tau_{2n}], n \in \mathbb{Z}, \\ 4 + |\sin \sqrt{7}t|, & t \in (\tau_{2n}, \tau_{2n+1}], n \in \mathbb{Z}, \end{cases}
$$

$$
\gamma_1(t) = \begin{cases} 1/4 + |\sin \sqrt{3}t|/4, & t \in (\tau_{2n-1}, \tau_{2n}], n \in \mathbb{Z}, \\ 1/4 + |\sin \sqrt{5}t|/4, & t \in (\tau_{2n}, \tau_{2n+1}], n \in \mathbb{Z}, \end{cases}
$$

$$
\gamma_2(t) = \begin{cases} |\sin \sqrt{5}t|/2, & t \in (\tau_{2n-1}, \tau_{2n}], n \in \mathbb{Z}, \\ |\sin \sqrt{7}t|/2, & t \in (\tau_{2n}, \tau_{2n+1}], n \in \mathbb{Z}. \end{cases}
$$

It is easy to verify that $\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2 \in AP_T(\mathbb{R})$. Moreover, we can get that $\sigma = \inf_{k\in\mathbb{Z}}(\tau_k - \tau_{k-1}) \geq 1/2$ and

$$\alpha^- = 10, \|\alpha\| = 11, \beta_1^- = 3, \|\beta_1\| = 4, \beta_2^- = 4, \|\beta_2\| = 5, \|\gamma_1\| = \|\gamma_2\| = 1/2.$$

Then we have

$$K = \frac{\|\beta_1\| + \|\beta_2\|}{\alpha^-} + \frac{\|\nu\|}{1 - e^{-\alpha^-\sigma}} \leq \frac{9}{10} + \frac{1/4}{1 - e^{-10/2}} < 1.2,$$

$$\underline{K}' = \frac{(1 - e^{-\alpha^-\sigma})^2}{\|\alpha\|} \sum_{j=1}^{2} \beta_j^- e^{-\|\gamma_j\|K} - \|\nu\| \geq \frac{(1 - e^{-5})^2}{11}(3\times e^{-0.6} + 4\times e^{-0.6}) - 1/4 > 0,$$

and

$$\|\beta_1\| + \|\beta_2\| = 9 < 10 = \alpha^-.$$

So by Remark 3.1 (ii), all the conditions of Theorem 3.1 is fulfilled, and then (3.2) has a unique positive solution $u \in AP_T(\mathbb{R})$ such that $0 < u(t) < K, t \in \mathbb{R}$.

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