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On Symmetric Bi-Multipliers of Incline Algebras

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Abstract

In this study, we introduce the notion of $*$ and $+$-symmetric bi-multipliers in incline algebras and research some related properties. Also, we define kernel of $*$ and $+$-symmetric bi-multipliers in incline algebras. Additionally, we state some properties of these $*$ and $+$-symmetric bi-multipliers in integral incline algebras.

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1. Introduction

Inclines are a generalization of both Boolean and fuzzy algebras, and a special type of a semiring. The notion of inclines is introduced and their applications are studied in [1]. Incline algebra and applications were studied by some authors in [2, 9]. It has both a semiring structure and a poset structure. Inclines can also be used to represent automa and other mathematical systems, in optimization theory, to study inequalities for nonnegative matrices of polynomials.

A partial multiplier on a commutative semigroup $(A, \cdot)$ has been introduced in [5] as a function F from a nonvoid subset D_F of A into A such that $F(x) \cdot y = x \cdot F(y)$ for all $x, y \in D_F$. It has been generalized to the partial multipliers on partially ordered sets in [7], [8]. The concept of multiplier of BE-algebras is given in [3] and obtained some properties of BE-algebras. They also introduced the simple multiplier and characterized the kernel of multipliers of BE-algebras. Later, the notion of multipliers is introduced in a hypersemilattice and some properties of multipliers are studied in [6].

In this paper the notion of $*$ and $+$-symmetric bi-multipliers in an incline algebra are given and properties of these multipliers are researched. Additionally, these definitions in an integral incline algebra are studied and related properties are given. Also, kernels and $Fix_f(R)$ of an incline algebra are characterized by $*$ and $+$ symmetric bi-multipliers.

2. Preliminaries

Definition 2.1. [1] An incline algebra is a non-empty set R with binary operations denoted by $+$ and $*$ satisfying the following axioms for all $x, y, z \in R$:

(RI) $x + y = y + x$,

(RII) $x + (y + z) = (x + y) + z$,

(RIII) $x * (y * z) = (x * y) * z$,

(RIV) $x * (y + z) = (x * y) + (x * z)$,

(RV) $(y + z) * x = (y * x) + (z * x)$,

(RVI) $x + x = x$,

(RVII) $x + (x * y) = x$,

(RVIII) $y + (x * y) = y$.

For convenience, we pronounce " $+$ " (resp." $*$ ") as addition (resp. multiplication). Moreover, Incline theory is based on semiring theory and lattice theory. Every distributive lattice is an incline. An incline is a distributive lattice (as a semiring) if and only if $x * x = x$ for all $x \in R$ ([3, Proposition (1.1.1)]).

Note that $x \leq y$ if and only if $x + y = y$ for all $x, y \in R$. It is easy to see that $\leq$ is a partial order on R and that for any $x, y \in R$, the element $x + y$ is the least upper bound of x, y. We say that $\leq$ is induced by operation $+$. In an incline algebra R, the following properties hold.

(1) $x * y \leq x$ and $x * y \leq y$ for all $x, y \in R$.

(2) $y \leq z$ implies $x * y \leq x$ and $y * x \leq z * x$ for any $x, y, z \in R$.

(3) If $x \leq y$, $a \leq b$, then $x + a \leq y + b$, $x * a \leq y * b$.

Furthermore, an incline algebra R is said to be commutative if $x * y = y * x$ for all $x, y \in R$.

A subincline of an incline R is a nonempty subset M of R which is closed under addition and multiplication.

An ideal in an incline R is a subincline $M \subseteq R$ such that if $x \in M$ and $y \leq x$ then $y \in M$. An element 0 in an incline algebra R is a zero element if $x + 0 = x = 0 + x$ and $x * 0 = 0 * x = 0$ for any $x \in R$. An element 1 ($\neq$ zero element) in an incline algebra R is called multiplicative identity if for any $x \in R$, $x * 1 = 1 * x = x$. A non-zero element a in an incline algebra R with a zero element is said to be a left (resp. right) zero divisor if there exists a non-zero element $b \in R$ such that $a * b = 0 (resp.\ b * a = 0)$. A zero divisor is an element of R which is both a left zero divisor and a right zero divisor. An incline algebra R with a multiplicative identity 1 and a zero element 0 is called an integral incline if it has no zero divisors.

3. $*$-Symmetric Bi-Multiplier Of An Incline Algebra

The following definition introduces the notion of $*$-symmetric bi-multiplier for an incline algebra. In what follows, let R denote an incline algebra unless otherwise specified.

Definition 3.1. Let R be an incline algebra. A mapping $f(.,.) : R \times R \to R$ is called *symmetric* if $f(x, y) = f(y, x)$ for all $x, y \in R$.

Definition 3.2. Let R be an incline algebra and let $f(.,.) : R \times R \to R$ be a symmetric mapping. We call f a *$*$-symmetric bi-multiplier* on R if it satisfies;

$$f(x, y * z) = f(x, y) * z \text{ for all } x, y, z \in R.$$

Example 3.1. *Let $R = \{a, b, c, d, f\}$, and we define the sum "$+$" and product "$*$" on R as follows:*

Then $(R, +, *)$ is an incline but not a distributive lattice.

$+$	a	b	c	d	f
a	a	a	a	a	a
b	a	b	a	b	b
c	a	a	c	c	c
d	a	b	c	d	d
f	a	b	c	d	f

$*$	a	b	c	d	f
a	a	b	c	d	f
b	b	b	d	d	f
c	c	f	c	f	f
d	d	f	d	f	f
f	f	f	f	f	f

The mapping $f(.,.) : R \times R \to R$ will be defined by

$$f(x,y) = \begin{cases} f, & \text{if } x \neq y, \\ a, & \text{if } x = y = a, \\ b, & \text{if } x = y = b, \\ c, & \text{if } x = y = c, \\ d, & \text{if } x = y = d, \\ f, & \text{if } x = y = f, \end{cases}$$

Then we can see that f is a $*$-symmetric bi-multiplier on R.

Example 3.2. *Let R be the same incline algebra defined in Example 3.1. The mapping $f(.,.) : R \times R \to R$ will be defined by*

$$f(x,y) = f$$

for all $x, y \in R$ is a $*$-symmetric bi-multiplier.

Example 3.3. *Let R be an incline algebra. The mapping $f(.,.) : R \times R \to R$ will be defined by*

$$f(x,y) = x * y$$

for all $x, y \in R$ is a $*$-symmetric bi-multiplier.

Proposition 3.3. *Let R be an incline algebra with a multiplicative identity and f be the $*$-symmetric bi-multiplier on R. Then the followings hold for all $x, y, z \in R$:*

i) $f(x,y) = f(x,1) * y$,

ii) If $f(x,1)=1$ *then* $f(x,y)=y$*,*

iii) $f(x,y*z)\leq f(x,y)+z$*.*

Proof : Let R be an incline algebra with a multiplicative identity and f be the $*$-symmetric bi-multiplier on R.
i) Let x,y be elements in R. Then we have

$$\begin{aligned}f(x,y)&=f(x,1*y)\\&=f(x,1)*y\end{aligned}$$

Therefore, $f(x,y)=f(x,1)*y$.
ii) It is clear from i).
iii) Let x,y,z be elements in R. Then we have

$$f(x,y*z)=f(x,y)*z\leq f(x,y)$$

And also we have $f(x,y)*z\leq z$. Therefore, we have $f(x,y)\leq f(x,y)+z$.

Proposition 3.4. *Every $*$-symmetric bi-multiplier is regular.*

Proof : Let f be a $*$-symmetric bi-multiplier on an incline algebra with a zero element. Then we have

$$\begin{aligned}f(0,0)&=f(0,x*0)\\&=f(0,x)*0\\&=0\end{aligned}$$

Let f be a $*$-symmetric bi-multiplier of an incline algebra R and a be fixed element in R. Define a set $Fix_f(R)=\{x\in R|f(a,x)=x\}$ for all $x\in R$.

Proposition 3.5. *Let f be a $*$-symmetric bi-multiplier on an incline algebra. If $x\in Fix_f(R)$ then $x*y\in Fix_f(R)$ for all $x,y\in R$.*

Proof : Let f be a $*$-symmetric bi-multiplier on an incline algebra and $x\in Fix_f(R)$. Then we have $f(a,x)=x$.

$$\begin{aligned}f(a,x*y)&=f(a,x)*y\\&=x*y\end{aligned}$$

Therefore, we have $x*y\in Fix_f(R)$.
Let f be a $*$-symmetric bi-multiplier of an incline algebra R with a zero element. Define a set $Ker_f=\{x\in R|f(0,x)=0\}$ for all $x\in R$.

Proposition 3.6. *Let f be a $*$-symmetric bi-multiplier of R with a zero element. If $x\in Ker_f$ then $x*y\in Ker_f$ for all $y\in R$.*

Proof : Let f be a $*$-symmetric bi-multiplier of R with a zero element and $x\in Ker_f$. Then we have $f(0,x)=0$. Let $y\in R$ then

$$\begin{aligned}f(0,x*y)&=f(0,x)*y\\&=0*y=0\end{aligned}$$

So, we get $x*y\in Ker_f$.

Theorem 3.7. *Let f be a $*$-symmetric bi-multiplier of an integral incline R with a zero element. If $f(0, x+y) = f(0, x) + f(0, y)$ for all $x, y \in R$ then Ker_f is an ideal.*

Proof : Let f be a $*$-symmetric bi-multiplier of an integral incline R with a zero element. And assume that $f(0, x+y) = f(0, x) + f(0, y)$ for all $x, y \in R$. Let $x \in R$ and $y \neq 0 \in Ker_f$ such that $x \leq y$. Then we have

$$\begin{aligned} 0 = f(0, y) &= f(0, y + (x * y)) \\ &= f(0, y) + f(0, x * y) \\ &= 0 + f(0, x) * y \\ &= f(0, x) * y \end{aligned}$$

Since R is an integral incline algebra we have $f(0, x) = 0$ or $y = 0$. But $y \neq 0 \in Ker_f$ so $f(0, x) = 0$. Therefore, we have Ker_f is an ideal.

4. +-Symmetric Bi-Multiplier of An Incline Algebra

The following definition introduces the notion of +-symmetric bi-multiplier for an incline algebra. In what follows, let R denote an incline algebra unless otherwise specified.

Definition 4.1. Let R be an incline algebra. A mapping $f(.,.) : R \times R \to R$ is called *symmetric* if $f(x, y) = f(y, x)$ for all $x, y \in R$.

Definition 4.2. Let R be an incline algebra and let $f(.,.) : R \times R \to R$ be a symmetric mapping. We call f a *+-symmetric bi-multiplier* on R if it satisfies;

$$f(x, y + z) = f(x, y) + z \text{ for all } x, y, z \in R.$$

Example 4.1. *Let R be an incline algebra. The mapping $f(.,.) : R \times R \to R$ will be defined by*

$$f(x, y) = x + y$$

is a +-symmetric bi-multiplier.

Proposition 4.3. *Let R be an incline algebra with a zero element and f be the +-symmetric bi-multiplier on R. Then we have the followings for all $x, y \in R$.*

i) $y \leq f(x, y)$

ii) $f(x, 0) + y \leq f(x, y)$

iii) If $f(0, 0) = 0$ then $x \leq f(0, x)$.

Proof :

i) Let R be an incline algebra and f be the +-symmetric bi-multiplier on R and $x, y \in R$. Then by using the definition of +-symmetric bi-multiplier and (RVI) we get

$$\begin{aligned} f(x,y) &= f(x,y+y) \\ &= f(x,y)+y \end{aligned}$$

Therefore $y \leq f(x,y)$.

ii) Let R be an incline algebra with a zero element and f be the +-symmetric bi-multiplier on R. Then we have

$$\begin{aligned} f(x,y) &= f(x,0+y) \\ &= f(x,0)+y \end{aligned}$$

Therefore $f(x,y) = f(x,0)+y$. Hence $f(x,0)+y \leq f(x,y)$.

iii) This is clear by ii).

Let f be a +-symmetric bi-multiplier of an incline algebra R and a be fixed element in R. Define a set $Fix_f(R) = \{x \in R | f(a,x) = x\}$ for all $x \in R$.

Proposition 4.4. *Let f be a +-symmetric bi-multiplier on an incline algebra. If $x \in Fix_f(R)$ then $x+y \in Fix_f(R)$ for all $y \in R$.*

Proof : Let f be a +-symmetric bi-multiplier on an incline algebra and $x \in Fix_f(R)$. Then we have $f(a,x) = x$.

$$\begin{aligned} f(a,x+y) &= f(a,x)+y \\ &= x+y \end{aligned}$$

Therefore, we have $x+y \in Fix_f(R)$.

Proposition 4.5. *Let f be a +-symmetric bi-multiplier on an incline algebra that is right cancellative. If $x+y \in Fix_f(R)$ and $y \in Fix_f(R)$ then $x \in Fix_f(R)$.*

Proof : Let f be a +-symmetric bi-multiplier on an incline algebra and $x+y \in Fix_f(R)$ and $y \in Fix_f(R)$. Then

$$\begin{aligned} f(a,x+y) &= f(a,x)+y \\ &= x+y \end{aligned}$$

Therefore we get $f(a,x)+y = x+y$. If R is additively right cancellative then we have $f(a,x) = x$. So $x \in Fix_f(R)$.

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