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Boundedness of Generalized Riesz Potentials on Commutative Hypergroups

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Abstract

The boundedness from Lebesgue space into certain Orlicz space is proven for generalized Riesz potential on commutative hypergroups.

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1 Introduction and Preliminary Notes

By the classical Hardy-Littlewood-Sobolev theorem, if $1 < p < \infty$ and $0 < \alpha p < n$, then a classical Riesz potential

$$R_\alpha f(x) = \int_{R^n} |x-y|^{\alpha-n} f(y)dy$$

is a bounded operator from $L^p(R^n)$ into $L^q(R^n)$, where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

In [1] and [6], generalized potential-type integral operators were considered and (p, q) properties of these operators were proved. In [12], [13], [14], [7], [8] the Hardy-Littlewood-Sobolev theorem was extended to Orlicz spaces for generalized fractional integrals. The analogues of the Hardy-Littlewood- Sobolev theorem were given for Riesz potentials on different hypergroups in [3], [4], [5], [2], [15] and on commutative hypergroups in [9], [10].

In this paper, we define generalized fractional integrals on commutative hypergroups and prove the analogue of Theorem 1.3 in [13] for the generalized Riesz potentials on commutative hypergroups.

Let $(K,*)$ be a hypergroup in the sense of Jewett (see [11]). By δ_x we denote a point measure at the point $x \in K$. The involution of $x \in K$ will be denoted by $x^{\sim}$. If $\delta_x * \delta_y = \delta_y * \delta_x$ for all $x, y \in K$, then the hypergroup K is called commutative. It is known that every commutative hypergroup K possesses a Haar measure which will be denoted by λ. That is, for every Borel measurable function f on K,

$$\int_K f(\delta_x * \delta_y) d\lambda(y) = \int_K f(y) d\lambda(y) \ \ (x \in K).$$

Define the generalized translation operators T^x, $x \in K$, by

$$T^x f(y) = \int_K f d(\delta_x * \delta_y)$$

for all $y \in K$. If K is a commutative hypergroup, then $T^x f(y) = T^y f(x)$ and the convolution of two functions is defined by

$$(f * g)(x) = \int_K T^x f(y) g(y^{\sim}) d\lambda(y).$$

Let $p > 0$. By $L^p(K,\lambda)$ denote a class of all λ-measurable functions $f : K \to (-\infty, +\infty)$ with $\|f\|_{L^p(K,\lambda)} = \left(\int_K |f(x)|^p d\lambda(x)\right)^{\frac{1}{p}} < \infty$.

A function $\Phi : [0,\infty] \to [0,\infty]$ is called an N-function if can be represented as $\Phi(r) = \int_0^r \phi(t)\, dt$, where $\phi : [0,\infty] \to [0,\infty]$ is a left continuous nondecreasing function such that $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$.

Let Φ is an N-function. Define the Orlicz space $L^\Phi(K,\lambda)$ to be the set of all locally integrable functions f in K for which $\int_K \Phi\left(\frac{|f(x)|}{\eta}\right) d\lambda(x) < \infty$ for some $\eta > 0$. Here $L^\Phi(K,\lambda)$ is equipped with the norm

$$\|f\|_\Phi = \inf\{\eta > 0 : \int_K \Phi\left(\frac{|f(x)|}{\eta}\right) d\lambda(x) \le 1\}.$$

For $\Phi(r) = r^p, 1 < p < \infty$, we have $L^\Phi(K,\lambda) = L^p(K,\lambda)$.

Let $(K,*)$ be a commutative hypergroup, with quasi-metric ρ, Haar measure λ and all balls $B(x,r) = \{y \in K : \rho(x,y) < r\}$ be λ-measurable and

$N \in (0,\infty)$. We will say Haar measure λ is upper Ahlfors N-regular at an identity, if there exists a constant $C>0$, not depending $r>0$, such that $\lambda B(e,r) \le Cr^N$, where e is an identity of hypergroup $(K,*)$.
For an increasing function $a:(0,\infty)\to(0,\infty)$, define

$$I_h f(x) = \int\limits_K T^x \left(\frac{h(\rho(e,y))}{\rho(e,y)^N} \right) f(y^\sim) d\lambda(y).$$

If $h(r) = r^\alpha, 0<\alpha<N$, then I_h is the Riesz potential of order α.
Define Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{r>0} \frac{1}{\lambda B(e,r)} \left(|f| * \chi_{B(e,r)} \right)(x)$$

on commutative hypergroup $(K,*)$ equipped with the quasi-metric ρ.
Now we formulate a main result of the paper.

Theorem 1.1 *Let $(K,*)$ be a commutative hypergroup, with the quasi-metric ρ and Haar measure λ, upper Ahlfors N-regular at an identity. Assume that $1<p<\infty$ and $h=h(r)$ is non-negative almost increasing function on $[0,\infty)$, $\frac{h(r)}{r^\lambda}$ is almost decreasing for some $0<\lambda<\frac{N}{p}$ and $\int\limits_0^1 \frac{h(t)}{t} dt < \infty$. If Hardy-Littlewood maximal operator $Mf(x)$ is bounded on $L^p(K,\lambda)$, then the operator I_h is bounded from $L^p(K,\lambda)$ into the Orlicz space $L^\Phi(K,\lambda)$, where the N-function is defined by its inverse $\Phi^{-1}(r) = \int\limits_0^r H\left(t^{-\frac{1}{N}}\right) t^{-\frac{1}{p'}} dt$, where $H(r) = \int\limits_0^r \frac{h(t)}{t} dt$.*

2 Proof of Theorem 1.1

We may suppose that $f(x) \ge 0$ and by the linearity of the operator I_h, it suffices to prove that $\|I_a f\|_\Phi \le C < \infty$ for $\|f\|_{L^p(K,\lambda)} \le 1$. Since Hardy-Littlewood maximal operator $Mf(x)$ is bounded on $L^p(K,\lambda)$, we have

$$\|Mf\|_{L^p(K,\lambda)} \le 1 \tag{1}$$

Split $I_h f(x)$ in the standard way

$$I_h f(x) = \int\limits_{B(e,r)} \frac{h(\rho(e,y))}{\rho(e,y)^N} T^x f(y^\sim) d\lambda(y)$$

$$+ \int\limits_{X\setminus B(e,r)} \frac{h(\rho(e,y))}{\rho(e,y)^N} T^x f(y^\sim) d\lambda(y) = \mathcal{A}_r(x) + \mathcal{B}_r(x).$$

Since $\frac{h(t)}{t^N}$ is almost decreasing, we have

$$\mathcal{A}_r(x) = \sum_{k=0}^{\infty} \int\limits_{2^{-k-1}r \le \rho(e,y) < 2^{-k}r} \frac{h(\rho(e,y))}{\rho(e,y)^N} T^x f(y^{\sim})\, d\lambda(y)$$

$$\le C \sum_{k=0}^{\infty} \frac{h\left(2^{-k-1}r\right)}{\left(2^{-k-1}r\right)^N} \int\limits_{2^{-k-1}r \le \rho(e,y) < 2^{-k}r} T^x f(y^{\sim})\, d\lambda(y) \le CMf(x) \sum_{k=0}^{\infty} h\left(2^{-k-1}r\right)$$

$$\le CMf(x) \sum_{k=0}^{\infty} \int\limits_{2^{-k-1}r}^{2^{-k}r} \frac{h(t)}{t} dt.$$

Therefore,

$$\mathcal{A}_r(x) \le CH(r)Mf(x), \quad H(r) = \int\limits_0^r \frac{h(t)}{t}\, dt. \tag{2}$$

By the Hölder inequality and the condition $\|f\|_{L^p(K,\lambda)} \le 1$, we obtain

$$\mathcal{B}_r(x) \le \left(\int\limits_{K \setminus B(e,r)} (T^x f(y^{\sim}))^p\, d\lambda(y) \right)^{\frac{1}{p}} \left(\int\limits_{K \setminus B(e,r)} \left(\frac{h(\rho(e,y))}{\rho(e,y)^N} \right)^{p'} d\lambda(y) \right)^{\frac{1}{p'}}$$

$$\le \left(\int\limits_{K \setminus B(e,r)} \left(\frac{h(\rho(e,y))}{\rho(e,y)^N} \right)^{p'} d\lambda(y) \right)^{\frac{1}{p'}}$$

$$= \left(\sum_{k=0}^{\infty} \int\limits_{2^k r \le \rho(e,y) < 2^{k+1}r} \left(\frac{h(\rho(e,y))}{\rho(e,y)^N} \right)^{p'} d\lambda(y) \right)^{\frac{1}{p'}}$$

$$\le C \left(\sum_{k=0}^{\infty} \left(\frac{h\left(2^k r\right)}{\left(2^k r\right)^N} \right)^{p'} \int\limits_{\rho(e,y) < 2^{k+1}r} d\lambda(y) \right)^{\frac{1}{p'}}$$

$$\le C \left(\sum_{k=0}^{\infty} \left(\frac{h\left(2^k r\right)}{\left(2^k r\right)^N} \right)^{p'} \left(2^{k+1}r\right)^N \right)^{\frac{1}{p'}} \le C \left(\sum_{k=0}^{\infty} \left(\frac{h\left(2^k r\right)}{\left(2^k r\right)^{\frac{N}{p}}} \right) \right)^{\frac{1}{p'}}$$

$$\le C \left(\sum_{k=0}^{\infty} \left(h\left(2^k r\right)\right)^{p'} \int\limits_{2^k r}^{2^{k+1}r} \left(\frac{1}{t^{\frac{N}{p}}} \right)^{p'} \frac{1}{t} dt \right)^{\frac{1}{p'}} \le C \left(\sum_{k=0}^{\infty} \int\limits_{2^k r}^{2^{k+1}r} \left(\frac{h(t)}{t^{\frac{N}{p}}} \right)^{p'} \frac{1}{t} dt \right)^{\frac{1}{p'}}$$

$$= C \left(\int\limits_r^{\infty} \left(\frac{h(t)}{t^{\frac{N}{p}}} \right)^{p'} \frac{1}{t} dt \right)^{\frac{1}{p'}} \le C \frac{h(r)}{r^{\beta}} \left(\int\limits_r^{\infty} \left(t^{\beta - \frac{N}{p}}\right)^{p'} t^{-1} dt \right)^{\frac{1}{p'}} \le C \frac{h(r)}{r^{\frac{N}{p}}}$$

Therefore

$$\mathcal{B}_r(x) \leq CH(r)r^{-\frac{N}{p}} \tag{3}$$

From (2) and (3), we have

$$I_h f(x) \leq C\left(Mf\left(x\right) + r^{-\frac{N}{p}}\right) H\left(r\right).$$

Then

$$I_h f(x) \leq C\left[Mf(x)r^{\frac{N}{p}} + 1\right]\Phi^{-1}\left(\frac{1}{r^N}\right) \tag{4}$$

by Theorem 4.9 in [8]. If we choose $r = [Mf(x)]^{-\frac{p}{N}}$, then the inequality (4) turns into $I_h f(x) \leq C\Phi^{-1}\left([Mf(x)]^p\right)$ and consequently, $\int\limits_K \Phi\left(\frac{I_h f(x)}{C}\right) d\lambda(x) \leq \int\limits_K [Mf(x)]^p d\lambda(x) \leq 1$, where we have used (1). Hence $\|I_h f\|_\Phi \leq C$, which completes the proof.

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