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On the k-Barycentric Olson Constant in $\prod_{i=1}^{m} \mathbb{Z}_p$

Henry Márquez

Departamento de Matemáticas, Universidad de Oriente
Cumaná 6101-A, Apartado 245, Venezuela

José Salazar

Departamento de Matemáticas, Universidad de Oriente
Cumaná 6101-A, Apartado 245, Venezuela

Felicia Villarroel

Departamento de Matemáticas, Universidad de Oriente
Cumaná 6101-A, Apartado 245, Venezuela

Abstract

Given the finite abelian group G and a positive integer k, the k-barycentric Olson constant, denoted by $BO(k, G)$, is the smallest positive integer q such that every set of cardinality q in G contains a subset with k elements $\{a_1, a_2, \ldots, a_k\}$ that satisfies the following property: $\sum_{i=1}^{k} a_i = ka_j$ for some $j \in \{1, 2, \ldots, k\}$. Such a subset with k elements is called k-barycentric and the element a_j corresponding to the set is called k-barycenter. The k-barycentric Olson constant has been studied in cyclic finite abelian groups, however this constant has not been studied in non-cyclic finite abelian groups. In this paper some results are shown for the k-barycentric Olson constant in non-cyclic finite abelian groups $\prod_{i=1}^{m} \mathbb{Z}_p$ where p is a prime number and an algorithm for calculating $BO(k, \prod_{i=1}^{m} \mathbb{Z}_2)$ is presented.

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1 Introduction

Let G be a finite abelian group of order n and S be a sequence of elements in G, i.e., the repetition of elements is allowed and the order of placing of the elements is not considered. Given $S \subseteq G$ a sequence or set, let $|S|$ be the length or cardinality of S and $\sum(S) = \{\sum_{a\in A} a : \emptyset \neq A \subseteq S\}$. If $\sum_{a\in A} a = 0$ one says that A is zero-sum.

The first known result on zero-sum problems is called by Erdös Prehistoric Lema: Let G be an abelian group of order n. Then any sequence of n elements contains a zero-sum subsequence. Erdös *et al* [4] have the following theorem: Any sequence of $2n-1$ elements in an abelian group of order n contains a zero-sum n-subsequence. This result is the fundamental basis in the development of the research area called Zero-Sum Problems, which is immersed in the field of Combinatorial Theory and, therefore, uses many results of the theory and basic tools.

Weighted sequences, i.e., sequences constituted by terms of the form $w_i a_i$ where the a_i's are elements of G and the coefficients or weights are positive integers, appear initially in the Caro conjecture [1].

Hamidoune [6] proved partially that conjecture which was later demonstrated by Grynkiewicz [5]. The fact that Hamidoune partially prove the Caro conjecture, allowed Ordaz to introduce the concept of k-barycentric sequences: Let G be an abelian group of order $n \geq 2$ and A be a finite set with $|A| \geq 2$. A sequence $f : A \to G$ is barycentric if there exists $a \in A$ verifying $\sum_A f = |A| f(a)$. The element $f(a)$ is called barycenter. When $|A| = k$ we talk about k-barycentric sequences and when f is injective we can use the expression k-barycentric set.

For example, in the non-cyclic finite abelian group

$$\prod_{i=1}^{3} \mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \{000, 010, 100, 110, 001, 011, 101, 111\}$$

of order 8, the set $\{000, 010, 100, 110\}$ is 4-barycentric because

$$000 \oplus 010 \oplus 100 \oplus 110 = 000 = 4 \cdot (000)$$

and 000 is barycenter.

However, the set $\{000, 010, 100, 001\}$ is not 4-barycentric since

$$000 \oplus 010 \oplus 100 \oplus 001 = 111 \notin \{4 \cdot (000), 4 \cdot (010), 4 \cdot (100), 4 \cdot (001)\}.$$

The definition of barycentric sequences initiates the barycentric problems. It is important to note that barycentric sequences generalize the sequences of zero-sum when their lengths are a multiple of the order of the group where they are defined.

Barycentric sequences are studied in [2, 3, 7, 8, 9, 10]. In [2] and [3] the study of these sequences begins. In [7] some observations of the barycentric-sum problems on cyclic groups are shown. In [8] the k-barycentric Olson constant, $BO(k, G)$, is studied; it is defined as the smallest positive integer q such that every q-set in G contains a k-barycentric subset. An algorithmic method based on matrix theory to calculate $BO(k,\mathbb{Z}_n)$ for $3 \leq n \leq 23$ and $3 \leq k \leq n$ is given in [9]. An algorithmic method based on the theory of orbits for the calculation of the constant $\mathbb{Z}_n$, for $3 \leq n \leq 12$ and $3 \leq k \leq n$ is given in [10]. In this paper some results of the k-barycentric Olson constant in noncyclic finite abelian groups $\prod_{i=1}^{m}\mathbb{Z}_2$ are shown and an algorithm for the calculation of $BO(k, \prod_{i=1}^{m}\mathbb{Z}_2)$ is presented. Some results of this constant for the non-cyclic finite abelian group $\prod_{i=1}^{m}\mathbb{Z}_p$ with $p \geq 3$ a prime number are also included.

2 Preliminary Results

Proposition 2.1. *If $m \geq 2$ is an integer and k is an even integer such that $3 \leq k \leq 2^m$, then $\forall x \in \prod_{i=1}^{m}\mathbb{Z}_2 : k \cdot x = e$, where $e = (0, , ..., 0)$.*

Proof. Let $k = 2w$ for some $w \in \mathbb{N}$ and $x \in \prod_{i=1}^{m}\mathbb{Z}_2$. Then

$$k \cdot x = (2w) \cdot x = 2 \cdot (w \cdot x) = 2 \cdot y = e,$$

where $w \cdot x = y \in \prod_{i=1}^{m}\mathbb{Z}_2$. Therefore, $\forall x \in \prod_{i=1}^{m}\mathbb{Z}_2 : k \cdot x = e$. □

Proposition 2.2. *If $m \geq 2$ is an integer and $x, y \in \prod_{i=1}^{m}\mathbb{Z}_2$ such that $x \neq y$, then $x \oplus y \neq e$.*

Proof. Let $x, y \in \prod_{i=1}^{m}\mathbb{Z}_2$ such that $x \neq y$.

$$x \neq y \Rightarrow \exists j \in \{1, 2, ..., m\} \ni x_j \neq y_j \Rightarrow x_j + y_j = 1 \Rightarrow x \oplus y \neq e.$$

Therefore, $x \oplus y \neq e$. □

As an immediate consequence of propositions 2.1 and 2.2, we have

Corollary 2.3. *If $m \geq 2$, then $BO(2, \prod_{i=1}^{m}\mathbb{Z}_2)$ does not exist.*

Proposition 2.4. *If $m \geq 2$ is an integer, then $\bigoplus_{x\in\prod_{i=1}^{m}\mathbb{Z}_2} x = e$.*

Proof. Let $\prod_{i=1}^{m}\mathbb{Z}_2$ be the noncyclic finite abelian group of even order 2^m. Among the 2^m elements of $\prod_{i=1}^{m}\mathbb{Z}_2$, there are exactly 2^{m-1} that have a 1 in i-th component, for $i = 1, 2, ..., m$. Since 2^{m-1} is even, then the sum of the 2^m elements results 0 in every component. Therefore, $\bigoplus_{x\in\prod_{i=1}^{m}\mathbb{Z}_2} x = e$. □

Proposition 2.5. *If $m \geq 2$ is an integer and k is an odd integer such that $3 \leq k \leq 2^m - 1$, then $\forall x \in \prod_{i=1}^{m} \mathbb{Z}_2 : k \cdot x = x$.*

Proof. Let $k = 2w + 1$ for some $w \in \mathbb{N}$ and $x \in \prod_{i=1}^{m} \mathbb{Z}_2$. Then

$$k \cdot x = (2w + 1) \cdot x = (2w) \cdot x \oplus x = 2 \cdot (w \cdot x) \oplus x = e \oplus x = x.$$

Therefore, $\forall x \in \prod_{i=1}^{m} \mathbb{Z}_2 : k \cdot x = x$. $\square$

Proposition 2.6. *If $m \geq 2$ is an integer and $A = \prod_{i=1}^{m} \mathbb{Z}_2 - \{z\}$, then $\bigoplus_{a \in A} a = z$.*

Proof. Let $A = \prod_{i=1}^{m} \mathbb{Z}_2 - \{z\}$ and $z^{'}$ the opposite of z. Since $z^{'} = z$ and $\bigoplus_{x \in \prod_{i=1}^{m} \mathbb{Z}_2} x = e$, then

$$e = \bigoplus_{x \in \prod_{i=1}^{m} \mathbb{Z}_2} x = \bigoplus_{a \in A} a \oplus z \Rightarrow \bigoplus_{a \in A} a \oplus z = e \Rightarrow \bigoplus_{a \in A} a = e \oplus z^{'}$$

$$\Rightarrow \bigoplus_{a \in A} a = z^{'} \Rightarrow \bigoplus_{a \in A} a = z.$$

Therefore, $\bigoplus_{a \in A} a = z$. $\square$

Proposition 2.7. *If $m \geq 2$ is an integer and $p \geq 3$ is a prime number, then $\forall x \in \prod_{i=1}^{m} \mathbb{Z}_p : (p^m - 1) \cdot x = x^{'}$, where $x^{'}$ is the opposite of x.*

Proof. Let $x \in \prod_{i=1}^{m} \mathbb{Z}_p$. Then

$$e = p^m \cdot x = (p^m - 1) \cdot x \oplus x \Rightarrow (p^m - 1) \cdot x \oplus x = e \Rightarrow (p^m - 1) \cdot x = x^{'}.$$

Therefore, $\forall x \in \prod_{i=1}^{m} \mathbb{Z}_p : (p^m - 1) \cdot x = x^{'}$. $\square$

Proposition 2.8. *If $m \geq 2$ is an integer and $A = \prod_{i=1}^{m} \mathbb{Z}_p - \{z\}$ with $p \geq 3$, then $\bigoplus_{a \in A} a = z^{'}$.*

Proof. Let $z \in \prod_{i=1}^{m} \mathbb{Z}_p$ and $A = \prod_{i=1}^{m} \mathbb{Z}_p - \{z\}$. Then

$$e = \bigoplus_{x \in \prod_{i=1}^{m} \mathbb{Z}_p} x = \bigoplus_{a \in A} a \oplus z \Rightarrow \bigoplus_{a \in A} a \oplus z = e \Rightarrow \bigoplus_{a \in A} a = z^{'}.$$

Therefore, $\bigoplus_{a \in A} a = z^{'}$. $\square$

3 Main Results

Given $m \geq 2$ an integer, let $\prod_{i=1}^{m} \mathbb{Z}_2$ be the noncyclic finite abelian group of even order 2^m and $\mathbb{Z}_n$ be the cyclic finite abelian group of order n. When $\prod_{i=1}^{m} \mathbb{Z}_2$ and $\mathbb{Z}_n$ have the same order, it happens that for some values of k, the constants $BO(k, \prod_{i=1}^{3} \mathbb{Z}_2)$ and $BO(k, \mathbb{Z}_n)$ are different. For example, when $k = 8$, one has that $BO(8, \mathbb{Z}_8)$ does not exist but $BO(8, \prod_{i=1}^{3} \mathbb{Z}_2) = 8$ and when $k = 7$, one has that $BO(7, \mathbb{Z}_8) = 7$ but $BO(7, \prod_{i=1}^{3} \mathbb{Z}_2)$ does not exist. Similar results apply more generally to the 2^m-barycentric and $(2^m - 1)$-barycentric Olson constants, for every integer $m \geq 2$.

Theorem 3.1. *If $m \geq 2$ is an integer, then $BO(2^m, \prod_{i=1}^{m} \mathbb{Z}_2) = 2^m$.*

Proof. Since $\prod_{i=1}^{m} \mathbb{Z}_2$ has 2^m elements, it is the only set to consider. By proposition 2.4, the sum of its elements is e, which, by proposition 2.1, can be obtained as $2^m \cdot x$ for any x in the set. Therefore, $BO(2^m, \prod_{i=1}^{m} \mathbb{Z}_2) = 2^m$. □

As an immediate consequence we have

Corollary 3.2. *Let $m \geq 2$ be an integer and k be an even integer, $2 \leq k \leq 2^m$, and A be a subset of $\prod_{i=1}^{m} \mathbb{Z}_2$ with cardinality k. Then A is k-barycentric if and only if $\bigoplus_{a \in A} a = e$.*

Theorem 3.3. *If $m \geq 2$ is an integer, then $BO(2^m - 1, \prod_{i=1}^{m} \mathbb{Z}_2)$ does not exist.*

Proof. Let $(\prod_{i=1}^{m} \mathbb{Z}_2, \oplus)$ be the noncyclic finite abelian group of order 2^m and $A = \prod_{i=1}^{m} \mathbb{Z}_2 - \{z\}$ be any subset of odd cardinality $2^m - 1$ from $\prod_{i=1}^{m} \mathbb{Z}_2$. By Proposition 2.5, it holds that $\forall a \in A : (2^m - 1) \cdot a = a$ and by Proposition 2.6, we have that $\bigoplus_{a \in A} a = z$.

Then

$$\forall a \in A : \bigoplus_{a \in A} a \neq (2^m - 1) \cdot a,$$

i.e., $\prod_{i=1}^{m} \mathbb{Z}_2$ does not contain any $(2^m - 1)$-barycentric subset. Therefore, $BO(2^m - 1, \prod_{i=1}^{m} \mathbb{Z}_2)$ does not exist. □

As an immediate consequence we have

Corollary 3.4. *Let $m \geq 2$ be an integer and k be an odd integer, $3 \leq k \leq 2^m - 3$, and A be a subset of $\prod_{i=1}^{m} \mathbb{Z}_2$ with cardinality k. Then A is k-barycentric if and only if there exists $x \in A$ such that $\bigoplus_{a \in A} a = x$.*

Theorem 3.5. *If $m \geq 2$ is an integer, then $BO(2^m - 2, \prod_{i=1}^{m} \mathbb{Z}_2)$ does not exist.*

Proof. Let $(\prod_{i=1}^{m}\mathbb{Z}_2, \oplus)$ be the noncyclic finite abelian group of order 2^m and $A = \prod_{i=1}^{m}\mathbb{Z}_2 - \{y, z\}$ be any subset of even cardinality $2^m - 2$ from $\prod_{i=1}^{m}\mathbb{Z}_2$ with $y \neq z$. By Proposition 2.1, we have that $\forall a \in A : (2^m - 2) \cdot a = e$.

Moreover,

$$e = \bigoplus_{x\in\prod_{i=1}^{m}\mathbb{Z}_2} x = \bigoplus_{a\in A} a \oplus y \oplus z \Rightarrow \bigoplus_{a\in A} a \oplus y \oplus z = e \Rightarrow \bigoplus_{a\in A} a = (y \oplus z)^{'}$$

$$\Rightarrow \bigoplus_{a\in A} a = y^{'} \oplus z^{'} \Rightarrow \bigoplus_{a\in A} a = y \oplus z \neq e$$

$$\Rightarrow \bigoplus_{a\in A} a \neq e.$$

Thus

$$\forall a \in A : \bigoplus_{a\in A} a \neq (2^m - 2) \cdot a,$$

i.e., $\prod_{i=1}^{m}\mathbb{Z}_2$ does not contain any $(2^m - 2)$-barycentric subset. Therefore, $BO(2^m - 2, \prod_{i=1}^{m}\mathbb{Z}_2)$ does not exist. □

Theorem 3.6. *If $m \geq 2$ is an integer, then $BO(3, \prod_{i=1}^{m}\mathbb{Z}_2)$ does not exist.*

Proof. Suppose $BO(3, \prod_{i=1}^{m}\mathbb{Z}_2) = q$ with $3 \leq q \leq 2^m$, e.i., every q-set A in $\prod_{i=1}^{m}\mathbb{Z}_2$ contains a 3-barycentric subset $B = \{x, y, z\}$ con $x \neq y \neq z$. By Corollary 3.4, there exists $b \in B$ such that $x \oplus y \oplus z = b$.

Since in $\prod_{i=1}^{m}\mathbb{Z}_2$ it holds the cancellation law, then

$$y \oplus z = e \qquad \vee \qquad x \oplus z = e \qquad \vee \qquad x \oplus y = e,$$

which contradicts the Proposition 2.2. This proves the theorem. □

Theorem 3.7. *Let $m \geq 2$ be an integer and k be an even integer such that $4 \leq k \leq 2^m - 4$. If $BO(k, \prod_{i=1}^{m}\mathbb{Z}_2) = q$ with $q > k$, then $BO(k+1, \prod_{i=1}^{m}\mathbb{Z}_2) = q$.*

Proof. Suppose $BO(k, \prod_{i=1}^{m}\mathbb{Z}_2) = q$ with $q > k$, e.i., every q-set A in $\prod_{i=1}^{m}\mathbb{Z}_2$ contains a k-barycentric subset B. By Corollary 3.2, $\bigoplus_{b\in B} b = e$.

Let $C = B \cup \{x\}$ be subset of odd cardinality $k+1$ in A, where $x \in A - B$ and $k + 1 \leq q$.

As

$$\bigoplus_{c\in C} c = \bigoplus_{b\in B} b \oplus x = e \oplus x = x,$$

then by Corollary 3.4, C is a $(k + 1)$-barycentric subset of q-set $A \subseteq \prod_{i=1}^{m}\mathbb{Z}_2$, e.i., $BO(k + 1, \prod_{i=1}^{m}\mathbb{Z}_2) \leq q$.

Suppose $BO(k+1, \prod_{i=1}^m \mathbb{Z}_2) = q_1$ with $q_1 < q$, e.i., every q_1-set A_1 in $\prod_{i=1}^m \mathbb{Z}_2$ contains a $(k+1)$-barycentric subset B_1. By Corollary 3.4, there exists $x \in B_1$ such that $\bigoplus_{b \in B_1} b = x$.

Let $C_1 = B_1 - \{x\}$ be subset of even cardinality k in $A_1 \subseteq \prod_{i=1}^m \mathbb{Z}_2$, where $k < k+1 \leq q_1 < q$.

As

$$\bigoplus_{c \in C_1} c = \bigoplus_{b \in B_1 - \{x\}} b = e;$$

then by Corollary 3.4, C_1 is a k-barycentric subset of q_1-set A_1, e.i., $BO(k, \prod_{i=1}^m \mathbb{Z}_2) \leq q_1$ with $q_1 < q$, which contradicts the hypothesis of the theorem. This proves the theorem. □

Given $m \geq 2$ an integer and $p \geq 3$ a prime number, let $\prod_{i=1}^m \mathbb{Z}_p$ be the noncyclic finite abelian group of odd order p^m and $\mathbb{Z}_n$ be the cyclic finite abelian group of order n. When $\prod_{i=1}^m \mathbb{Z}_p$ and $\mathbb{Z}_n$ have the same order, it occurs that for some values of k, the Olson constants for $\prod_{i=1}^m \mathbb{Z}_p$ and $\mathbb{Z}_n$ are equal. For example, when $k = 9$, one has $BO(9, \mathbb{Z}_9) = 9$ and $BO(9, \prod_{i=1}^2 \mathbb{Z}_3) = 9$, and when $k = 8$, one has that $BO(8, \mathbb{Z}_9)$ and $BO(8, \prod_{i=1}^2 \mathbb{Z}_3)$ do not exist. Similar results apply more generally to the p^m-barycentric and $(p^m - 1)$-barycentric Olson constants, for every integer $m \geq 2$ and for all $p \geq 3$.

Theorem 3.8. *If $m \geq 2$ is an integer, then $BO(p^m, \prod_{i=1}^m \mathbb{Z}_p) = p^m$.*

Proof. Since $\prod_{i=1}^m \mathbb{Z}_p$ has p^m elements, it is the only set to consider. To calculate the sum of its elements, it is enough to observe that, since p is not even, the opposite of each element is different from itself, excepting only the neutral e. Hence,

$$\bigoplus_{x \in \prod_{i=1}^m \mathbb{Z}_p} = e.$$

On the other hand, since p^m is a multiple of p, then

$$\forall x \in \prod_{i=1}^m \mathbb{Z}_p : p^m \cdot x = e.$$

Thus

$$\exists x \in \prod_{i=1}^m \mathbb{Z}_p : \bigoplus_{x \in \prod_{i=1}^m \mathbb{Z}_p} = p^m \cdot x,$$

implying that $BO(p^m, \prod_{i=1}^m \mathbb{Z}_p) = p^m$. □

Theorem 3.9. *If $m \geq 2$ is an integer, then $BO(p^m - 1, \prod_{i=1}^m \mathbb{Z}_p)$ does not exist.*

Proof. Let's see that no subset of cardinality $p^m - 1$ is $(p^m - 1)$-barycentric. Indeed, by Proposition 2.8, the sum of the elements of the subset is equal to the opposite of the element that does not belong to the subset; while, by proposition 2.7, the product of $p^m - 1$ times any element of the subset is the opposite of that element. By the property of uniqueness of the opposite in $\prod_{i=1}^m \mathbb{Z}_p$, the opposite of the element outside the subset cannot be equal to the opposite of an element in the subset. Thus, there is no $(p^m - 1)$-barycentric set. Therefore, $BO(p^m - 1, \prod_{i=1}^m \mathbb{Z}_p)$ does not exist. $\square$

Proposition 3.10. *If $m \geq 2$ is an integer, $p \geq 3$ is a prime number and k is a odd number such that $3 \leq k \leq p^m$, then $BO(k, \prod_{i=1}^m \mathbb{Z}_p)$ exists.*

Proof. Since, by Theorem 3.8, we have that $BO(p^m, \prod_{i=1}^m \mathbb{Z}_p) = p^m$, let consider k to be an odd number such that $3 \leq k \leq p^m - 2$. Define the subset A of cardinality k in $\prod_{i=1}^m \mathbb{Z}_p$ by $A = B \cup \{e\}$ where B is a subset of even cardinality $k-1$ in $\prod_{i=1}^m \mathbb{Z}_p$ consisting of $\frac{k-1}{2}$ elements $x \neq e$ jointly with their respective opposites $x^{'} \neq e$, so that $\bigoplus_{b \in B} b = e$.

Then
$$\bigoplus_{a \in A} a = \bigoplus_{b \in B} b \oplus e = e \oplus e = e$$

Moreover, $\exists e \in A : k \cdot e = e$.

Thus
$$\exists x \in A : \bigoplus_{a \in A} a = k \cdot x,$$
implying that $\prod_{i=1}^m \mathbb{Z}_p$ contains a k-barycentric subset, i.e., $BO(k, \prod_{i=1}^m \mathbb{Z}_p)$ exists. $\square$

Proposition 3.11. *If $m \geq 2$ is an integer, $p \geq 3$ is a prime number and k is an even multiple of p such that $3 \leq k \leq p^m$, then $BO(k, \prod_{i=1}^m \mathbb{Z}_p)$ exists.*

Proof. Given k an even multiple of p such that $3 \leq k \leq p^m$, let A be a subset of cardinality k in $\prod_{i=1}^m \mathbb{Z}_p$ formed by $\frac{k}{2}$ elements $x \neq e$ jointly with their $\frac{k}{2}$ opposites $x^{'} \neq e$. Then $\bigoplus_{a \in A} a = e$.

Since k is a multiple of p and $A \subset \prod_{i=1}^m \mathbb{Z}_p$, then $\forall x \in A : k \cdot x = e$.

Thus
$$\exists x \in A : \bigoplus_{a \in A} a = k \cdot x,$$
implying that $\prod_{i=1}^m \mathbb{Z}_p$ contains a k-barycentric subset, i.e., $BO(k, \prod_{i=1}^m \mathbb{Z}_p)$ exists. $\square$

As an immediate consequence of propositions 3.10 and 3.11, we have

Corollary 3.12. *If $m = 2$ and k is a number such that $3 \leq k \leq 9$ and $k \neq 8$, then $BO(k, \prod_{i=1}^2 \mathbb{Z}_3)$ exists.*

4 Method for calculation of $BO(k,\prod_{i=1}^{m}\mathbb{Z}_2)$

Suppose you want to calculate $BO(k,\prod_{i=1}^{m}\mathbb{Z}_2)$ with $\mathbb{Z}_2=\{0,1\}$, $m\geq 2$ a positive integer and $3\leq k\leq 2^m$. First, you compute the number $j=2^m$, which represents the order of the set $\prod_{i=1}^{m}\mathbb{Z}_2$. Then you check whether the conditions of the theorems are true: (1) If $m\geq 2$, then $BO(3,\prod_{i=1}^{m}\mathbb{Z}_2)$ does not exist. (2) If $m\geq 2$, then $BO(2^m-2,\prod_{i=1}^{m}\mathbb{Z}_2)$ does not exist. (3) If $m\geq 2$, then $BO(2^m-1,\prod_{i=1}^{m}\mathbb{Z}_2)$ does not exist. (4) If $m\geq 2$, then $BO(2^m,\prod_{i=1}^{m}\mathbb{Z}_2)=2^m$. Otherwise, you construct the set $\prod_{i=1}^{m}\mathbb{Z}_2$ and compute the number of combinations without repetition of elements from $\prod_{i=1}^{m}\mathbb{Z}_2$, taken k by k, using the formula $\binom{j}{k}=\frac{j!}{k!(j-k)!}$.

You form, one by one, the $\binom{j}{k}$ sets of cardinality k in $\prod_{i=1}^{m}\mathbb{Z}_2$ and check at once whether they are k-barycentric or not. If they are all k-barycentric, the method ends, and you get $BO(k,\prod_{i=1}^{m}\mathbb{Z}_2)=k$. Otherwise, you assign to a variable q the value $k+1$, you compute the number of combinations without repetition of the j elements of $\prod_{i=1}^{m}\mathbb{Z}_2$ taken q by q, which is $\binom{j}{q}=\frac{j!}{q!(j-q)!}$, and the number of combinations without repetition of the q elements of the corresponding sets taken k by k, which is $\binom{q}{k}=\frac{q!}{k!(q-k)!}$.

You form, one by one, the $\binom{j}{q}$ sets of cardinality q in $\prod_{i=1}^{m}\mathbb{Z}_2$ and for each of them, you form, one by one, the $\binom{q}{k}$ subsets of cardinality k and check whether they are k-barycentric or not. If all sets of cardinality q contain some k-barycentric subset, then the method ends and you get $BO(k,\prod_{i=1}^{m}\mathbb{Z}_2)=q$. Otherwise, q is increased by 1 and the process continues. The final level is reached when q exceeds j, in which case the method has ended and $BO(k,\prod_{i=1}^{m}\mathbb{Z}_2)$ does not exist

5 Application of the method

Let's see how you get $BO(4,\prod_{i=1}^{3}\mathbb{Z}_2)=5$ using the method. First, you compute the number $j=2^m=2^3=8$, which is the order of the group $\prod_{i=1}^{3}\mathbb{Z}_2$. Since $4\notin\{3,6,7,8\}$, the conditions of the theorems 3.1, 3.3, 3.5 and 3.6 are not met and you construct the set

$$\prod_{i=1}^{3}\mathbb{Z}_2=\{000,010,100,110,001,011,101,111\}.$$

The number of its subsets with 4 elements is $\binom{8}{4}=70$. You have to check each one of these 70 subsets to see if it is 4-barycentric, but when you arrive at $\{000,010,100,101\}$, you get a subset that fails to be 4-barycentric, because

$$000\oplus 010\oplus 100\oplus 101=011\notin\{4\cdot(000),4\cdot(010),4\cdot(100),4\cdot(101)\}=\{e\}.$$

Then, you does not need to check the remaining subsets, you know that the Olson constant is not 4. So you consider $q = k + 1 = 5$; there are $\binom{8}{5} = 56$ sets of cardinality 5 in $\prod_{i=1}^{3} \mathbb{Z}_2$, each of which allows for $\binom{5}{4} = 5$ subsets of cardinality 4. It so happens that for each of the 56 sets, you find one of its subsets to be 4-barycentric (these 56 findings are not reported here for lack of space). Thus, the method ends, and you get $BO(4, \prod_{i=1}^{3} \mathbb{Z}_2) = 5$.

6 Algorithm for the calculation of $BO(k, \prod_{i=1}^{m} \mathbb{Z}_2)$

Begin

Input: $m \geq 2$ and $k \geq 2$ positive integers

Assign to the variable j the value 2^m

If $k = 3 \quad \vee \quad k = j - 2 \quad \vee \quad k = j - 1$, then

Out: $BO(k, \prod_{i=1}^{m} \mathbb{Z}_2)$ does not exist

Else

If $k = j$, then

Out: $BO(k, \prod_{i=1}^{m} \mathbb{Z}_2) = k$

Else

Construct the set $\prod_{i=1}^{m} \mathbb{Z}_2 = \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\text{m factors}}$

Assign to the variable d the value $\frac{j!}{k!(j-k)!}$

Initialize the variable r with 1

While $r \leq d$ do

Determine the k-subset S_r in $\prod_{i=1}^{m} \mathbb{Z}_2$

If S_r is not k-barycentric,

Assign to the variable r the value $d + 2$

Else

Assign to the variable r the value $r + 1$

End If

End While

If $r = d + 1$, then

Out: $BO(k, \prod_{i=1}^{m}\mathbb{Z}_2) = k$

Else

Assign to the variable q the value $k + 1$

While $q \leq j$ do

Assign to the variable r the value 1

Assign to the variable d the value $\frac{j!}{q!(j-q)!}$

Assign to the variable e the value $\frac{q!}{k!(q-k)!}$

While $r \leq d$ do

Determine the q-set C_r in $\prod_{i=1}^{m}\mathbb{Z}_2$

Assign to the variable l the value 1

While $l \leq e$ do

Determine the k-subset S_l of C_r

If S_l is k-barycentric, then

Assign to the variable l the value $e + 2$

Else

Assign to the variable l the value $l + 1$

End If

End While

If $l = e + 2$, then

Assign to the variable r the value $r + 1$

Else

Assign to the variable r the value $d + 2$

End If

End While

If $r = d + 1$,then

Out: $BO(k, \prod_{i=1}^{m} \mathbb{Z}_2) = q$

If k is even,then

Out: $BO(k + 1, \prod_{i=1}^{m} \mathbb{Z}_2) = q$

End If

Assign to the variable j the value $j + 2$

Else

Assign to the variable q the value $q + 1$

End If

End While

If $q = j + 1$, then

Out: $BO(k, \prod_{i=1}^{m} \mathbb{Z}_2)$ does not exist

End If

End If

End If

End If

End Algorithm.

7 Exact Values of $BO(k, \prod_{i=1}^{m} \mathbb{Z}_2)$

The manual procedure to calculate some values of the $BO(k, \prod_{i=1}^{m} \mathbb{Z}_2)$ is long and tedious, and many times humanly impossible to obtain. This reason led us to develop and program in MuPAD an algorithm that calculates these values; MuPAD is a program designed to assist in performing mathematical calculations and graphs. Program execution threw the values obtained manually and other securities; which we show in the following table:

Value of m	**Value of k**	$BO(k,\prod_{i=1}^{m}\mathbb{Z}_2)$
2	3	does not exist
	4	4
3	3	does not exist
	4	5
	5	5
	6	does not exist
	7	does not exist
	8	8
4	3	does not exist
	4	7
	5	7
	6	10
	7	10
	8	11
	9	11
	10	13
	11	13
	12	13
	13	13
	14	does not exist
	15	does not exist
	16	16
5	3	does not exist
	26	29
	27	29
	28	29
	29	29
	30	does not exist
	31	does not exist
	32	32

Table 1: Exact Values of $BO=(k,\prod_{i=1}^{m}\mathbb{Z}_2)$, for $m=2,3,4,5$.

The values of $BO(k,\prod_{i=1}^{5}\mathbb{Z}_2)$ that are missing in table 1 could not be calculated due to insufficient memory on the computer. What follows now is to use high computer computations to obtain new values this constant. Then analyze it to try to obtain other algebraic results that will allow us improve the algorithm.

We propose the following:

Conjecture 7.1. *If $m\geq 2$ be an integer and k be an integer such that $4\leq k\leq 2^m-3$, then $BO(k,\prod_{i=1}^{m}\mathbb{Z}_2)$ exists.*

Conjecture 7.2. *Let $m \geq 2$ be an integer and k be an integer such that $4 \leq k \leq 2^m - 3$. If $BO(k, \prod_{i=1}^{m} \mathbb{Z}_2) = q_1$ and $BO(k+1, \prod_{i=1}^{m} \mathbb{Z}_2) = q_2$, then $q_1 \leq q_2$.*

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