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The Ternary Macaulay-Lex Ideals Generated by Monomials of the Form $x^u y^v z^w (w \leq 1)$

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Abstract

In this paper, we determine all ternary Macaulay-Lex ideals generated by monomials of the form $x^u y^v z^w (w \leq 1)$.

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Let $S = k[x_1, \cdots, x_n]$ be a polynomial ring over a field k graded by $deg(x_i) = 1$ for all i. Macaulay [7] showed that for every graded ideal there exists a lexicographic ideal with the same Hilbert function. Lexicographic ideals are defined combinatorially and have extremal Betti numbers [2], [6], [11].

Let I be a monomial ideal, $W = S/I$. We say that a graded ideal in W is lexifiable if there exists a lexicographic ideal in W with the same Hilbert function. We call I and W Macaulay-Lex if every graded ideal in W is lexifiable. By Gröbner basis theory, if every monomial ideal in W is lexifiable, then W is Macaulay-Lex. In the following, we only consider monomial ideals. Macaulay's theorem [7] says that (0) is a Macaulay-Lex ideal, Clements-Lindström's theorem [3] says that $R = (x_1^{e_1}, \cdots, x_n^{e_n})$ is a Macaulay-Lex ideal if $e_1 \leq \cdots \leq e_n \leq +\infty$, Mermin-Peeva's theorem [10] contains the above results. Mermin-Murai [9] prove that $I = \sum_{j=1}^{r} (x_{j,1}, \cdots, x_{j,n_j})^2$ is a Macaulay-Lex ideal for $x_{k,l} > x_{r,s}$ if $l > s$ or $l = s$ and $k < r$. Mermin's theorem [8] characterizes the monomial regular sequences which are Macaulay-Lex. All Macaulay-Lex ideals of $k[x,y]$ are determined in [4]. All Macaulay-lex ideals of $k[x,y,z]$ that do not contain any x^m are determined in [5]. Some Macaulay-Lex ideals of the form $(x_1^{e_1}, x_1^{t_1}x_2^{e_2}, x_1^{t_1}x_2^{t_2}x_3^{e_3}, \cdots, x_1^{t_1}\cdots x_{n-1}^{t_{n-1}}x_n^{e_n})$ are determined in [1]. In this paper, we determine all ternary Macaulay-Lex ideals generated by x^s and monomials of the form $x^u y^v z^w (w \leq 1)$.

Let S_p be the set of all monomials of degree p in S. We order the variables $x_1 > \cdots > x_n$ and denote $>$ the homogeneous lexicographic order on the monomial: for $f = \prod_{i=1}^{n} x_i^{a_i}, g = \prod_{i=1}^{n} x_i^{b_i} \in S_p$, $f > g$ if there exists m such that $a_m > b_m$ and $a_i = b_i (1 \leq i < m)$. Let $M = \{x_1, \cdots, x_n\}$. For $S = k[x,y,z]$, denote $x_1 = x$, $x_2 = y$, $x_3 = z$. Let I_p be the set of all monomials of degree p in I, $|L|$ be the cardinality of a set L. Denote $W_p = S_p \backslash I_p$ as a subset of S_p. Let $C_p \subseteq B_p \subseteq S_p$, C_p is called a lex-segment of length d in B_p if C_p contains exactly the first d monomials of B_p, L_p is called the lex-segment of C_p in B_p if L_p is a lex-segment of length $\mid C_p \mid$ in B_p. The empty set $\emptyset$ is a lex-segment of any set. A monomial ideal I of S is called lexicographic ideal or simply lex ideal if I_p is a lex-segment in S_p for every p. Denote $f = x^a y^b z^c = (a,b,c)$, $deg(f) = a+b+c$, $deg_x(f) = a$. For $1 \leq i \leq n$, let $A_{pi}(t) = \{f \in W_p \mid deg_{x_i}(f) \geq t\}$, $A_p(t) = A_{p1}(t)$. For $B_p \subseteq S_p$, $1 \leq i \leq n$, let $B_{pi}(t) = \{f \in B_p \mid deg_{x_i}(f) = t\}$, $B_p(t) = B_{p1}(t)$. If g divides f, denote $g \mid f$. For $B \subseteq S_p$, $C \subseteq S_q$, denote $BC = \{fg \mid f \in B, g \in C\}$, $\max(B)$ be the lex-greatest element of B, $\min(B)$ be the lex-smallest element of B.

Proposition 1. [10] Let I be a monomial ideal of S, then I is a Macaulay-Lex ideal of S if and only if $\forall p \geq 1$, $J_p \subseteq W_p$, let L_p be the lex-segment of J_p in W_p, then $|ML_p \backslash I| = |ML_p \backslash I_{p+1}| \leq |MJ_p \backslash I_{p+1}| = |MJ_p \backslash I|$.

Proposition 2. [10] Let $f_i \in T = k[x_1, \cdots, x_{n-1}] (n \geq 2)$, $(f_i \mid 1 \leq i \leq \alpha)$ be a Macaulay-Lex ideal of T, then $(f_i \mid 1 \leq i \leq \alpha)$ is a Macaulay-Lex ideal of $S = k[x_1, \cdots, x_n]$.

Proposition 3. [10] If L is a lex ideal of S, R is a Macaulay-Lex ideal of S, then $L + R$ is a Macaulay-Lex ideal of S.

Proposition 4. [4] Let

$$I = (x^{a_0}, x^{a_1}y^{b_1}, \cdots, x^{a_r}y^{a_r})$$

be an ideal of $S = k[x,y]$; $r \geq 0$; $a_0 > a_1 > \cdots > a_r \geq 0$; $0 = b_0 < b_1 < \cdots < b_r$, then I is a Macaulay-Lex ideal of S if and only if I satisfies the following conditions

(4.1) If $b_1 = 1$, then $a_i + b_i \geq a_0 (2 \leq i \leq r)$; If $b_1 \geq 2$, then $a_i + b_i \geq a_0 (1 \leq i \leq r)$;

(4.2) $a_{i-1} + b_i \leq a_i + b_{i+1} (1 \leq i < r)$

(4.3) $\forall p \in \{a_i + b_i \mid 0 \le i \le r\} \cup \{+\infty\}$, let

$$X_p = \{i \mid 0 \le i \le r; p > a_i + b_i\}; X_{+\infty} = \{0, 1, \cdots, r\}$$

$$X_p = \{j(1) < j(2) < \cdots < j(u)\}$$

If $u \ge 3$, then

$$a_{j(v-1)} + b_{j(v)} \le a_{j(v)} + b_{j(v+1)} (2 \le v < u)$$

Definition 1. Let I be a monomial ideal of $k[x, y, z]$, we say that I satisfies the condition (ML3) for p, if $\mid ML_p \backslash I \mid \le \mid MJ_p \backslash I \mid$ holds for every $J_p \subseteq W_p$ and its lex-segment L_p in W_p.

Proposition 5. [5] Let $I = (x^s) + (g_i \mid 1 \le i \le \alpha), g_i = x^{u_i} y^{v_i} z^{w_i}$, $s = u_0 > u_i > u_{i+1}$, $v_i \le v_{i+1}$, $w_i \le w_{i+1} \le 1$, $1 \le v_i + w_i < v_{i+1} + w_{i+1} (1 \le i < \alpha)$. If I satisfies the following conditions (5.1), (5.2), then the condition (5.3) and (5.4) are equivalent. If I satisfies all the following conditions, then I is a Macaulay-Lex ideal of $S = k[x, y, z]$.

(5.1) Let $q = \min\{u_i + v_i + w_i \mid 1 \le i \le \alpha, v_i + w_i \ge 2\}$, then $q \ge s$

(5.2) $u_{i-1} + v_i + w_i \le u_i + v_{i+1} + w_{i+1} (1 \le i < \alpha)$

(5.3) Let $q \le p + 1 \le u_{\alpha-1} + v_\alpha + w_\alpha - 2$, then I satisfies the condition (ML3) for p

(5.4) Let $q \le p + 1 \le u_{\alpha-1} + v_\alpha + w_\alpha - 2$, $\psi(x) = (x + 1)(x + 2)$,

$$X_{p+1} = \{0\} \cup \{i \mid 1 \le i \le \alpha; p+1 \ge u_i + v_i + w_i\} = \{j(0) < j(1) < \cdots < j(\beta)\}$$

$\theta(t) = u_{j(t)} + v_{j(t+1)} + w_{j(t+1)} - p - 2 (0 \le t < \beta)$, $\theta(\beta) = +\infty$.

If $\theta(\sigma) \ge 1$, then $\theta(i) \ge 1 (\sigma \le i \le \beta)$.

If $\beta \ge 2$ and $\theta(\beta - 2) \ge 1$, let σ be the smallest one such that $\theta(\sigma) \ge 1$,

$$G = \{(a, b, c, d, \delta, \pi) \mid \sigma \le a < c < d \le \beta; b \ge a; 0 \le \pi < \theta(b);$$

$$0 \le \delta < \min\{\theta(c - 1), \theta(d)\}; \delta + \sum_{t=c}^{d-1} \theta(t) = \pi + \sum_{t=a}^{b-1} \theta(t)\}$$

then for every $(a, b, c, d, \delta, \pi) \in G$, the following inequality holds

$$\psi(p - u_{j(d)} - v_{j(c)} - w_{j(c)} + \delta) - \sum_{t=c}^{d} \psi(p - u_{j(t)} - v_{j(t)} - w_{j(t)})$$

$$\le \psi(p - u_{j(b)} + \pi) - \psi(p - u_{j(a)}) - \sum_{t=a+1}^{b} \psi(p - u_{j(t)} - v_{j(t)} - w_{j(t)})$$

Definition 2. Let I be a monomial ideal of $S = k[x, y, z]$, x^s be a minimal generator of I, denote $z(I) = \min\{a \mid x^a z^b \text{ is a minimal generator of } I\}$.

Lemma 6. Let $I = (g_i \mid 1 \le i \le \alpha)$ be a Macaulay-Lex ideal of $S = k[x_1, \cdots, x_n](n \ge 2)$, $g_i = (u_{i1}, \cdots, u_{in})(1 \le i \le \alpha)$ be minimal generators of I, $q = \min\{\sum_{j=1}^{n} u_{ij} \mid 1 \le i \le \alpha; \sum_{j=2}^{n} u_{ij} \le 1\}$, then $\sum_{j=1}^{n} u_{ij} \ge q(1 \le i \le \alpha)$.

Proof. Let $u_{im} \ge 1$. If $deg(g_i) < q$, let $f = x_m^{-1} g_i$, $p = deg(g_i) - 1$, $J_p = \{f\}$, then $L_p = \{x_1^p\}$, $x_1^p x_j \notin I(1 \le j \le n)$, $x_m f \in I$, hence

$$| ML_p \backslash I | = | \{x_1^p x_j \mid 1 \le j \le n\} | = n > n-1$$

$$= | \{x_j f \mid 1 \le j \le n, j \ne m\} | \ge | MJ_p \backslash I |$$

a contradiction, so $deg(g_i) \ge q$.

Theorem 7. Let I be an ideal generated by monomials of the form $x^u y^v z^w (w \le 1)$, x^s be a minimal generator of I, $r = z(I)$, $\Omega = \{f \mid f = (a, b, c)$ is a minimal generator of $I, a < r\} \ne \emptyset$. If I is a Macaulay-Lex ideal of $S = k[x, y, z]$, then Ω is one of the following three sets:

$$\{x^{u_i} y^{v_i} z^{w_i} \mid 1 \le i \le m\} \qquad (7.1)$$

$$\{x^{u_i} y^{v_i} z^{w_i} \mid 1 \le i \le m\} \cup \{x^{r-1} y^b z\} (w_1 = 0) \quad (7.2)$$

$$\{x^{a_i} y^{b_i} z \mid 1 \le i \le m\} \cup \{x^{r-1} y^v\} \quad (7.3)$$

where $u_i > u_{i+1}$, $v_i \le v_{i+1}$, $w_i \le w_{i+1} \le 1$, $v_i + w_i < v_{i+1} + w_{i+1}$, $a_i > a_{i+1}$, $b_i < b_{i+1} (1 \le i < m)$.

Proof. If $\Omega = \{x^{u_i} y^{v_i} \mid 1 \le i \le \alpha\}$ or $\{x^{u_i} y^{v_i} z \mid 1 \le i \le \alpha\}$, then Ω is the set (7.1).

Let $\Omega = \{g_i \mid 1 \le i \le \alpha\} \cup \{h_i \mid 1 \le i \le \pi\}$, $g_i = x^{u_i} y^{v_i} z$, $h_i = x^{a_i} y^{b_i}$, $u_i > u_{i+1}$, $v_i < v_{i+1} (1 \le i < \alpha)$, $a_i > a_{i+1}$, $b_i < b_{i+1} (1 \le i < \pi)$. If $v_1 = 0$, then $g_1 = x^{u_1} z$, $r = z(I) \le u_1 < r$, a contradiction, so $v_1 \ge 1$. Similarly, $b_1 \ge 1$.

Case 1. $u_1 \le r - 2$.

First, we prove that $a_\pi > u_1$. If $a_\pi \le u_1$, let a_i be the greatest one such that $a_i \le u_1$. Since h_i does not divide g_1, we have $b_i \ge v_1 + 1 \ge 2$. Let $f_2 = (u_1, b_i - 1, 0)$, $p = u_1 + b_i - 1$, then $h_i \mid y f_2$, $g_1 \mid z f_2$, hence $y f_2, z f_2 \in I$. If $j < i$, then $a_j > u_1$, h_j does not divide f_2. If $j \ge i$, then $b_j \ge b_i$, h_j does not divide f_2, hence $f_2 \notin I$. Let $f_1 = (u_1 + 1, 0, b_i - 2)$. Since $v_1 \ge 1$ and $b_1 \ge 1$, g_t and h_j do not divide $z f_1$, so $f_1, z f_1 \notin I$. Let $B = \{f \in W_p \mid f > f_1\}$.

If $b_i \ge 3$, then $f_3 = (u_1 + 1, b_i - 2, 0) > f_1$, $f_3 \in B \cup I$, $x f_2 = y f_3 \in MB \cup I$. Let $J_p = B \cup \{f_2\}$, then $L_p = B \cup \{f_1\}$, $| MJ_p \backslash I | = | MB \backslash I |$. $\forall f \in MB$, $deg_x(f) \ge u_1 + 2$ or $deg_y(f) \ge 1$, so $z f_1 \ne f$, $z f_1 \notin (MB) \cup I$,

$$| ML_p \backslash I | \ge | MB \backslash I | + | \{z f_1\} | = | MB \backslash I | + 1 > | MB \backslash I | = | MJ_p \backslash I |$$

a contradiction, so $b_i < 3$, $b_i = 2$, $v_1 = 1$, $i = 1, 2$.

Let $i = 1$, then $b_1 = 2$. If $r < s$, then $x^r z$ is a minimal generator of I. By Lemma 6, $u_1 + 2 = deg(g_1) \ge r + 1$, $u_1 \ge r - 1$, a contradiction. If $r = s$, then $u_1+2=deg(g_1)\geq s=r\geq u_1+2$, $u_1=s-2$, $g_1=x^{s-2}yz$. Similarly, $a_1+2=deg(h_1)\geq s$, $a_1\geq s-2=u_1\geq a_1$, $a_1=s-2$, $h_1=x^{s-2}y^2$. Let $f_1=x^{s-2}y$, $p=s-1$, $J_p=\{f_1\}$, then $L_p=\{x^p\}$,

$$|\ ML_p\backslash I\ |=|\ \{x^{s-1}y,x^{s-1}z\}\ |=2>1=|\ \{x^{s-1}y\}\ |=|\ MJ_p\backslash I\ |$$

a contradiction, so $i\neq 1$, $i=2$.

Let $i=2$, then $b_2=2$, $b_1=v_1=1$. By the choice of a_i, we have $a_1>u_1\geq a_2$. By Lemma 6, $u_1+2=deg(g_1)\geq a_1+1\geq u_1+2$, $u_1=a_1-1$, $a_2+2=deg(h_2)\geq a_1+1\geq a_2+2$, $a_2=a_1-1$, $g_1=(u_1,1,1)$, $h_1=(u_1+1,1,0)$, $h_2=(u_1,2,0)$. Let $f_2=(u_1,1,0)$, $p=a_1$, $J_p=\{f_2\}$, then $f_2\notin I$, $xf_2,yf_2,zf_2\in I$, $|\ MJ_p\backslash I\ |=0$, $L_p=\{x^p\}$. Since $s\geq r\geq a_1+1$, if $s>p+1=a_1+1$, then x^s does not divide x^{p+1}, $x^{p+1}\notin I$, $|\ ML_p\backslash I\ |\geq 1>|\ MJ_p\backslash I\ |$, a contradiction. If $s=a_1+1$, then $s=r=a_1+1>p$, x^s and g_i,h_i do not divide x^pz, hence $x^pz\notin I$, $|\ ML_p\backslash I\ |\geq 1>|\ MJ_p\backslash I\ |$, a contradiction, so $a_\pi>u_1$.

If $b_\pi\leq v_1$, then Ω is the set (7.1). Let $b_\pi\geq v_1+1\geq 2$.

If $b_\pi=2$, then $v_1=1$, $\pi=1,2$. If $\pi=2$, then $b_2=2$, $b_1=1$, $a_1>a_2>u_1$, $a_1\geq u_1+2=deg(g_1)\geq a_1+1$, a contradiction, so $\pi=1$, $b_1=2$. If $r<s$, then $a_1+1\geq u_1+2=deg(g_1)\geq r+1$, $a_1\geq r$, a contradiction, so $r=s$, $a_1+1\geq u_1+2=deg(g_1)\geq s\geq a_1+1$, $a_1=s-1$, Ω is the set (7.3).

Let $b_\pi\geq 3$. If $a_\pi<r-1$, let $f_2=y^{-1}h_\pi$, $p=a_\pi+b_\pi-1$, then $f_2\notin I$, $g_1\mid zf_2$, hence $yf_2,zf_2\in I$. Let $f_1=(a_\pi+1,0,b_\pi-2)$, $B=\{f\in W_p\mid f>f_1\}$, $J_p=B\cup\{f_2\}$, then $L_p=B\cup\{f_1\}$. Similar to the above, $xf_2\in MB\cup I$, $zf_1\notin MB\cup I$, $|\ ML_p\backslash I\ |\geq|\ \{zf_1\}\ |=1>0=|\ MJ_p\backslash I\ |$, a contradiction, so $a_\pi=r-1$, $\pi=1$, Ω is the set (7.3).

Case 2. $u_1=r-1$.

If $\alpha=1$, then Ω is the set (7.2). Let $\alpha\geq 2$, we prove that $a_\pi>u_2$. If $a_\pi\leq u_2$, let a_i be the greatest one such that $a_i\leq u_2$. Since h_i does not divide g_2, we have $b_i\geq v_2+1\geq v_1+2\geq 3$. Let $f_2=(u_2,b_i-1,0)$, $p=u_2+b_i-1$, $f_1=(u_2+1,0,b_i-2)$, $B=\{f\in W_p\mid f>f_1\}$, $J_p=B\cup\{f_2\}$, $L_p=B\cup\{f_1\}$, then $|\ ML_p\backslash I\ |>|\ MJ_p\backslash I\ |$ as above, a contradiction, so $a_\pi>u_2$.

If $b_\pi\leq v_2$, then Ω is the set (7.2). Let $b_\pi\geq v_2+1\geq 3$. Similar to the case $u_1\leq r-2$, $b_\pi\geq 3$, we have $a_\pi=r-1$, $\pi=1$, hence Ω is the set (7.3).

If $z(I)=s$ and $\Omega=\{x^{u_i}y^{v_i}z^{w_i}\mid 1\leq i\leq m\}$, then I is the ideal in Proposition 5. If $z(I)=s$ and $\Omega=\{x^{a_i}y^{b_i}z\mid 1\leq i\leq m\}\cup\{x^{r-1}y^v\}$, then I is the ideal in Proposition 5 and 8.

Proposition 8. Let $I=(x^s,x^{s-1}y^b)+(g_i\mid 1\leq i\leq\alpha)$, $g_i=x^{u_i}y^{v_i}z$, $s=u_0>u_i>u_{i+1}$, $1\leq v_i<v_{i+1}(1\leq i<\alpha)$, $b>v_1$. If $u_1=s-1$ and $\alpha\geq 2$, let $b>v_2$, then I is a Macaulay-Lex ideal of $S=k[x,y,z]$ if and only if $R=(x^s)+(g_i\mid 1\leq i\leq\alpha)$ is a Macaulay-Lex ideal of S.

Proof. ($\Leftarrow$): Let $D=(f\in S_{s+b-1}\mid f\geq x^{s-1}y^b)$, then D is a lex ideal of S. By Proposition 3, $I=R+D$ is a Macaulay-Lex ideal of S.

($\Rightarrow$): Let $V = S/R$, $J_p \subseteq V_p$, $J_p \neq \emptyset$, L_p be the lex-segment of J_p in V_p, $J_p \neq L_p$, we need prove that $\mid ML_p\backslash R \mid \leq \mid MJ_p\backslash R \mid$.

If $p+1 < s+b-1 = deg(x^{s-1}y^b)$, then $I_t = R_t$, $V_t = W_t (t \leq p+1)$, L_p is the lex-segment of J_p in W_p, hence

$$\mid ML_p\backslash R_{p+1} \mid = \mid ML_p\backslash I_{p+1} \mid \leq \mid MJ_p\backslash I_{p+1} \mid = \mid MJ_p\backslash R_{p+1} \mid \quad (8.1)$$

If $p+1 \geq s+b-1 \geq s+v_1$, then $p-s+1 \geq v_1$, $x^{s-1}y^{p-s+1}z \in R_{p+1}$. Let $B_p(t) = \{f \in V_p \mid deg_x(f) \geq t\}$, $g = (e_1, e_2, e_3) = \min(J_p)$, $h = (d_1, d_2, d_3) = \max(V_p\backslash J_p)$, $K_p = (J_p\backslash\{g\}) \cup \{h\}$, then $g < h$. If $e_3 = 0$, then $e_1 < s-1$, g_i does not divide yg, hence $yg \notin R$. Since g_i does not divide g, if $e_3 \geq 1$, then g_i does not divide zg, hence $zg \notin R$. Let $f = (c_1, c_2, c_3) \in J_p\backslash\{g\}$, then $f > g$. If $e_3 = 0$, then $c_1 > e_1$, $xf > yf > zf > yg$, $yg \notin M(J_p\backslash\{g\}) \cup R$, hence

$$\mid MJ_p\backslash R \mid \geq \mid M(J_p\backslash\{g\})\backslash R \mid + \mid \{yg\} \mid = \mid M(J_p\backslash\{g\})\backslash R \mid +1$$

If $e_3 \geq 1$, then $xf > yf > zf > zg$, $zg \notin M(J_p\backslash\{g\}) \cup R$, $\mid MJ_p\backslash R \mid \geq \mid M(J_p\backslash\{g\})\backslash R \mid +1$ holds also.

Let $d_1 = s-1$, then $xh \in R$. If $h = x^{s-1}y^{p-s+1}$, then $zh \in R$,

$$\mid MK_p\backslash R \mid \leq \mid M(J_p\backslash\{g\})\backslash R \mid + \mid \{yh\} \mid$$

$$= \mid M(J_p\backslash\{g\})\backslash R \mid +1 \leq \mid MJ_p\backslash R \mid \quad (8.2)$$

If $h \neq x^{s-1}y^{p-s+1}$, then $h_2 = yz^{-1}h > h$, $h_2 \in (J_p\backslash\{g\}) \cup R$, $yh = zh_2 \in M(J_p\backslash\{g\}) \cup R$, $\mid MK_p\backslash R \mid \leq \mid MJ_p\backslash R \mid$ holds also. If $\mid ML_p\backslash R \mid \leq \mid MK_p\backslash R \mid$, then $\mid ML_p\backslash R \mid \leq \mid MJ_p\backslash R \mid$. Replace J_p by K_p, namely, translate g to h. Repeat the above translation until $J_p = L_p \subseteq B_p(s-1)$ or $B_p(s-1) \subseteq J_p$. We can suppose that $B_p(s-1) \subseteq J_p$, then $B_p(s-1) \subseteq L_p$.

Since I is a Macaulay-Lex ideal of S and (x^{s-1}) is a lex ideal of S, $E = (x^{s-1}) + I = (x^{s-1}) + R$ is a Macaulay-Lex ideal of S. Let $U = S/E$, $C_p = J_p\backslash B_p(s-1)$, $D_p = L_p\backslash B_p(s-1)$, then D_p is the lex-segment of C_p in $U_p = V_p\backslash B_p(s-1)$, hence $\mid MD_p\backslash E \mid \leq \mid MC_p\backslash E \mid$. Next, we prove that

$$MC_p\backslash R \subseteq (MB_p(s-1)\backslash R) \cup (MC_p\backslash E)$$

Let $f = (e_1, e_2, e_3) \in C_p$, then $e_1 < s-1$. If $yf \in MC_p\backslash R$, then x^{s-1} does not divide yf, $yf \in MC_p\backslash E$. Similarly, if $zf \in MC_p\backslash R$, then $zf \in MC_p\backslash E$. Let $xf \in MC_p\backslash R$. If $e_1 \leq s-3$, then x^{s-1} does not divide xf, $xf \in MC_p\backslash E$. Let $e_1 = s-2$. Since $p \geq s+b-2 \geq s$, we have $e_2 \geq 1$ or $e_3 \geq 1$. If $e_2 \geq 1$, let $h = xy^{-1}f$, then $h \in B_p(s-1)$, $xf = yh \in MB_p(s-1)\backslash R$. Similarly, if $e_3 \geq 1$, then $xf \in MB_p(s-1)\backslash R$.

Since $R \subseteq E$, we have

$$MJ_p\backslash R = M(B_p(s-1) \cup C_p)\backslash R = (MB_p(s-1)\backslash R) \cup (MC_p\backslash R)$$

$$\subseteq (MB_p(s-1)\backslash R) \cup (MC_p\backslash E) \subseteq (MB_p(s-1)\backslash R) \cup (MC_p\backslash R) = MJ_p\backslash R$$

hence

$$MJ_p\backslash R = (MB_p(s-1)\backslash R) \cup (MC_p\backslash E)$$
$$ML_p\backslash R = (MB_p(s-1)\backslash R) \cup (MD_p\backslash E)$$

are disjoint unions,

$$| ML_p\backslash R | = | MB_p(s-1)\backslash R | + | MD_p\backslash E |$$

$$\leq | MB_p(s-1)\backslash R | + | MC_p\backslash E | = | MJ_p\backslash R | \quad (8.3)$$

If $z(I) = s$ and $\Omega = \{x^{u_i}y^{v_i}z^{w_i} \mid 1 \leq i \leq m\} \cup \{x^{r-1}y^bz\}(w_1 = 0)$, then I is the ideal in the following Proposition 9.

Proposition 9. Let $I = (x^s, x^{s-1}y^bz) + (g_i \mid 1 \leq i \leq \alpha)$, $g_i = x^{u_i}y^{v_i}z^{w_i}$, $s = u_0 > u_i > u_{i+1}$, $1 \leq v_i \leq v_{i+1}$, $w_i \leq w_{i+1} \leq 1, v_i + w_i < v_{i+1} + w_{i+1}(1 \leq i < \alpha)$, $v_1 > b \geq 1$, $w_1 = 0$, then I is a Macaulay -Lex ideal of S if and only if the following conditions hold

(9.1) Let $q = \min\{u_i + v_i + w_i \mid 1 \leq i \leq \alpha\}$, then $q \geq s$

(9.2) $E = (x^{s-1}) + (g_i \mid 1 \leq i \leq \alpha)$ is a Macaulay-Lex ideal of S

(9.3) If $\alpha \geq 2$, then $s + b < u_{\alpha-1} + v_\alpha + w_\alpha$

(9.4) If $p+1 < s+b$, then $R = (x^s)+(g_i \mid 1 \leq i \leq \alpha)$ satisfies the condition (ML3) for p

Proof. ($\Rightarrow$): By Lemma 6, (9.1) holds. Since (x^{s-1}) is a lex ideal of S and $E = (x^{s-1}) + I$, (9.2) holds.

If $p + 1 < s + b$, let $V = S/R$, $J_p \subseteq V_p$, L_p be the lex-segment of J_p in V_p, then $I_t = R_t$, $V_t = W_t(t \leq p + 1)$. Similar to (8.1), $| ML_p\backslash R | \leq | MJ_p\backslash R |$, (9.4) holds.

Next, we prove (9.3). If $u_1 = s - 1$, by (9.2) and (5.2),

$$s + b \leq s + v_1 - 1 = u_1 + v_1 < u_1 + v_2 + w_2 \leq u_{\alpha-1} + v_\alpha + w_\alpha$$

Let $u_1 < s - 1$. If $s + b \geq u_{\alpha-1} + v_\alpha + w_\alpha$, then $s + b \leq s - 1 + v_1 \leq u_1 + v_2 + w_2 \leq u_{\alpha-1} + v_\alpha + w_\alpha \leq s + b$, $s + b = s - 1 + v_1 = u_1 + v_2 + w_2$. Let $p = s + b - 2 = s + v_1 - 3 = u_1 + v_2 + w_2 - 2$,

$$f_2 = \begin{cases} (u_1 - 1, v_2 - 1, w_2) & v_2 > v_1 \\ (u_1 - 1, v_2, 0) & v_2 = v_1 \end{cases}$$

then $f_2 \notin I$. If $v_2 > v_1$, then $g_1 \mid xf_2$, $g_2 \mid yf_2$, $xf_2, yf_2 \in I$. If $v_2 = v_1$, then $w_2 = 1$, $g_1 \mid xf_2$, $g_2 \mid zf_2$, $xf_2, zf_2 \in I$. Let $J_p = \{f_2\}$, $f_1 = x^{s-1}y^{p-s+1}$, then $L_p = \{f_1\}$. Since $p - s + 2 = b < v_1 \leq v_i$, $x^{s-1}y^bz$ and $g_i(1 \leq i \leq \alpha)$ do not divide yf_1 and zf_1, $yf_1, zf_1 \notin I$, $| ML_p\backslash I | \geq 2 > 1 \geq | MJ_p\backslash I |$, a contradiction, so $s + b < u_{\alpha-1} + v_\alpha + w_\alpha$.

($\Leftarrow$): Let $J_p \subseteq W_p$, $J_p \neq \emptyset$, L_p be the lex-segment of J_p in W_p, we need prove that $\mid ML_p \backslash I \mid \leq \mid MJ_p \backslash I \mid$.

If $p+1 \geq s+b$, then $p-s+1 \geq b$, $x^{s-1}y^b z \mid x^{s-1}y^{p-s+1}z = f$, $f \in I_{p+1}(s-1) \neq \emptyset$. Let $g = \min(J_p)$. If $h = \max(W_p \backslash J_p) \in W_p(s-1)$, let $K_p = (J_p \backslash \{g\}) \cup \{h\}$, then $xh \in I$. Similar to (8.2), $\mid MK_p \backslash I \mid \leq \mid MJ_p \backslash I \mid$. Translate g to h until $J_p = L_p \subseteq A_p(s-1)$ or $A_p(s-1) \subseteq J_p$. We can suppose that $A_p(s-1) \subseteq J_p$, then $A_p(s-1) \subseteq L_p$. Let $C_p = J_p \backslash A_p(s-1)$, $D_p = L_p \backslash A_p(s-1)$, $V = S/E$, then D_p is the lex-segment of C_p in $V_p = W_p \backslash A_p(s-1)$. By (9.2), $\mid MD_p \backslash E \mid \leq \mid MC_p \backslash E \mid$. Similar to (8.3),

$$\begin{aligned} \mid ML_p \backslash I \mid &= \mid MA_p(s-1) \backslash I \mid + \mid MD_p \backslash E \mid \\ &\leq \mid MA_p(s-1) \backslash I \mid + \mid MC_p \backslash E \mid = \mid MJ_p \backslash I \mid \end{aligned} \tag{9.5}$$

If $p+1 < s+b$, then $I_t = R_t (t \leq p+1)$. Similar to (8.1), $\mid ML_p \backslash I \mid \leq \mid MJ_p \backslash I \mid$ holds.

If $z(I) = r < s$ and $\Omega = \emptyset$, then I is the ideal in the following Proposition 10.

Proposition 10. Let $I = (x^s, x^r z) + (h_i \mid 1 \leq i \leq m)$, $h_i = x^{a_i} y^{b_i}$, $s > r$, $s > a_i > a_{i+1}$, $1 \leq b_i < b_{i+1} (1 \leq i < m)$, $a_m \geq r$, $h_0 = x^s$, then I is a Macaulay-Lex ideal of $S = k[x,y,z]$ if and only if $D = (h_i \mid 0 \leq i \leq m)$ is a Macaulay-Lex ideal of $T = k[x,y]$.

Proof. ($\Rightarrow$): By Proposition 4, D is a Macaulay-Lex ideal of T if and only if $E = x^{-r}D$ is a Macaulay-Lex ideal of T, we need only prove that E is a Macaulay-Lex ideal of T. Let $V = T/E$, $J_p \subseteq V_p$, L_p be the lex-segment of J_p in V_p, we need prove that $\mid \{x,y\} L_p \backslash E \mid \leq \mid \{x,y\} J_p \backslash E \mid$. Let $f \in T_p$, then $x^{-r} h_i \mid f$ if and only if $h_i \mid x^r f (0 \leq i \leq m)$, so $f \to x^r f : V_p \to A_{p+r}(r)$ is a bijection preserving lex-order, $D_{p+r} = x^r E_p$, $x^r J_p \subseteq A_{p+r}(r)$, $x^r L_p$ is the lex-segment of $x^r J_p$ in $x^r V_p = A_{p+r}(r)$, hence $\mid M(x^r L_p) \backslash I \mid \leq \mid M(x^r J_p) \backslash I \mid$. Since $x^r z J_p \subseteq I$ and $x^r z L_p \subseteq I$, we have,

$$\begin{aligned} \mid \{x,y\} L_p \backslash E_{p+1} \mid &= \mid \{x,y\}(x^r L_p) \backslash (x^r E_{p+1}) \mid = \mid \{x,y\}(x^r L_p) \backslash D_{p+r+1} \mid \\ &= \mid M(x^r L_p) \backslash I_{p+r+1} \mid \leq \mid M(x^r J_p) \backslash I_{p+r+1} \mid = \mid \{x,y\} J_p \backslash E_{p+1} \mid \end{aligned}$$

($\Leftarrow$): Let $J_p \subseteq W_p$, $J_p \neq \emptyset$, L_p be the lex-segment of J_p in W_p, we need prove that $\mid ML_p \backslash I \mid \leq \mid MJ_p \backslash I \mid$. Let $q = \min\{a_i + b_i \mid 0 \leq i \leq m\}$.

If $p+1 < q$, let $R = (x^r z)$, then R is a Macaulay-Lex ideal of S, $I_t = R_t (t \leq p+1)$. Similar to (8.1), $\mid ML_p \backslash I \mid \leq \mid MJ_p \backslash I \mid$ holds.

Let $p+1 \geq q$. If $F_p = A_p(r) \backslash J_p \neq \emptyset$, let $g = \min(J_p)$, $h = \max(F_p)$, $K_p = (J_p \backslash \{g\}) \cup \{h\}$, then $zh \in I$, xh or $yh \in M(J_p \backslash \{g\}) \cup I$. If $g \notin A_p(r)$, similar to (8.2), $\mid MK_p \backslash I \mid \leq \mid MJ_p \backslash I \mid$. Translate g to h until $A_p(r) \subseteq J_p$ or $J_p \subseteq A_p(r)$. If $J_p \subseteq A_p(r)$, then $L_p \subseteq A_p(r)$,

$$\mid ML_p \backslash I \mid = \mid \{x,y\} L_p \backslash D \mid \leq \mid \{x,y\} J_p \backslash D \mid = \mid MJ_p \backslash I \mid$$

If $A_p(r) \subseteq J_p$, then $A_p(r) \subseteq L_p$. Let $E = (x^r)$, $C_p = J_p \backslash A_p(r)$, $D_p = L_p \backslash A_p(r)$, $V = S/E$, then E is a Macaulay-Lex ideal of S, D_p is the lex-segment of C_p in $V_p = W_p \backslash A_p(r)$, hence $\mid MD_p \backslash E \mid \leq \mid MC_p \backslash E \mid$. Similar to (8.3) and (9.5), $\mid ML_p \backslash I \mid \leq \mid MJ_p \backslash I \mid$.

Similar to Proposition 10, interchange y and z, we have

Proposition 11. Let $I = (x^s, x^r y) + (h_i \mid 1 \leq i \leq m)$, $h_i = x^{a_i} z^{b_i}$, $s > r$, $s > a_i > a_{i+1}$, $1 \leq b_i < b_{i+1} (1 \leq i < m)$, $a_m \geq r$, $h_0 = x^s$, then I is a Macaulay-Lex ideal of $S = k[x, y, z]$ if and only if $(h_i \mid 0 \leq i \leq m)$ is a Macaulay-Lex ideal of $k[x, z]$.

If $z(I) = r < s$ and $\Omega = \{x^{u_i} y^{v_i} z^{w_i} \mid 1 \leq i \leq m\}$, then I is the ideal in Proposition 12, 13 and Corollary 15.

Proposition 12. Let $I = (x^s, x^r z) + (g_i \mid 1 \leq i \leq \alpha)$, $g_i = x^{u_i} y^{v_i} z^{w_i}$, $s > r$, $s > u_i > u_{i+1}$, $1 \leq v_i \leq v_{i+1}$, $w_i \leq w_{i+1} \leq 1$, $1 \leq v_i + w_i < v_{i+1} + w_{i+1} (1 \leq i < \alpha)$, $g_0 = x^s$, $(u_0, v_0, w_0) = (s, 0, 0)$, $u_{\sigma-1} \geq r > u_\sigma$, $w_\sigma = 0$, $v_\sigma \geq 2$, then I is a Macaulay-Lex ideal of $S = k[x, y, z]$ if and only if I satisfies the following conditions

(12.1) Let $q = \min\{u_i + v_i + w_i \mid \sigma \leq i \leq \alpha\}$, then $q \geq r + 1$

(12.2) $Y = (g_i \mid 0 \leq i < \sigma) + (x^r y^{v_\sigma})$ is a Macaulay-Lex ideal of $T = k[x, y]$. Let $m = \min\{u_i + v_i \mid 0 \leq i < \sigma\}$, then $r + v_\sigma \geq m$.

(12.3) $(x^r) + (g_i \mid \sigma \leq i \leq \alpha) = (x^r) + I$ is a Macaulay-Lex ideal of S

(12.4) If $\alpha > \sigma$, then $m < u_{\alpha-1} + v_\alpha + w_\alpha$

(12.5) If $w_\alpha = 1$, let $\lambda = \min\{u_i + v_i + 1 \mid 1 \leq i \leq \alpha, w_i = 1\}$, $\mu = \min\{u_i + v_i \mid 0 \leq i < \alpha, w_i = 0\}$, then $\lambda \geq \mu$.

(12.6) If $m > p + 1 \geq q$, then $R = (x^{r+1}) + (g_i \mid \sigma \leq i \leq \alpha)$ satisfies the condition (ML3) for p

Proof. ($\Rightarrow$): By Lemma 6, (12.1) holds. Since (x^r) is a lex ideal of S, by Proposition 3, (12.3) holds.

Next, we prove (12.2). Similar to Proposition 10, let $E = x^{-r} Y$, $V = T/E$, $J_p \subseteq V_p$, L_p be the lex-segment of J_p in V_p, then

$$\mid \{x, y\} L_p \backslash E_{p+1} \mid = \mid \{x, y\}(x^r L_p) \backslash (x^r E_{p+1}) \mid = \mid \{x, y\}(x^r L_p) \backslash Y_{p+r+1} \mid$$

$$= \mid M(x^r L_p) \backslash I_{p+r+1} \mid \leq \mid M(x^r J_p) \backslash I_{p+r+1} \mid = \mid \{x, y\} J_p \backslash E_{p+1} \mid$$

If $u_{\sigma-1} > r$, then $x^r y^{v_\sigma}$ is a minimal generator of Y. By (4.1), $r + v_\sigma \geq s \geq m$. Let $u_{\sigma-1} = r$, then $r + v_\sigma = u_{\sigma-1} + v_\sigma > u_{\sigma-1} + v_{\sigma-1} \geq m$.

Next, we prove (12.4). Similar to (9.3), if $m \geq u_{\alpha-1} + v_\alpha + w_\alpha = p + 2$, let

$$f_2 = \begin{cases} (u_{\alpha-1} - 1, v_\alpha - 1, w_\alpha) & v_\alpha > v_{\alpha-1} \\ (u_{\alpha-1} - 1, v_\alpha, 0) & v_\alpha = v_{\alpha-1} \end{cases}$$

$J_p = \{f_2\}$, then $L_p = \{x^p\}$. For $0 \leq i < \sigma$, $u_i + v_i \geq m > p + 1$, g_i does not divide x^{p+1} and $x^p y$. For $i \geq \sigma$, $v_i \geq v_\sigma \geq 2$, g_i does not divide x^{p+1}

and x^py, so $x^{p+1}, x^py \notin I$, $\mid ML_p\backslash I \mid \geq 2 > 1 \geq \mid MJ_p\backslash I \mid$, a contradiction, so $m < u_{\alpha-1} + v_\alpha + w_\alpha$.

Next, we prove (12.5). If $\lambda < \mu$, let u_i be the greatest one such that $p+1 = \lambda = u_i + v_i + 1$, $J_p = A_p(u_i) \cap A_{p3}(1)$, then $L_p = A_p(u_i+1)$. Since $A_p(r) = \{x^ty^{p-t} \mid r \leq t \leq p\}$,

$$\mid MA_p(r)\backslash I \mid = \mid \{x,y\}A_p(r) \mid = \mid yA_p(r) \mid + \mid \{x^{p+1}\} \mid = \mid A_p(r) \mid +1$$

For $u_i < t < r$, $L_p(t) = S_p(t) = \{x^ty^{p-t-j}z^j \mid 0 \leq j \leq p-t\}$,

$$\mid \{y,z\}L_p(t)\backslash I \mid = \mid zL_p(t) \mid + \mid \{x^ty^{p-t+1}\} \mid = \mid L_p(t) \mid +1$$

For $u_i \leq t < r$, $J_p(t) = S_p(t)\backslash\{x^ty^{p-t}\}$. If $u_i < t < r$, then

$$\mid \{y,z\}J_p(t)\backslash I \mid = \mid zJ_p(t) \mid + \mid \{x^ty^{p-t}z\} \mid = \mid J_p(t) \mid +1 \quad (12.7)$$

If $t = u_i$, since $f_3 = y^{-1}g_i \in J_p(u_i)$, $yf_3 \in I$, we have

$$\mid \{y,z\}J_p(u_i)\backslash I \mid = \mid zJ_p(u_i) \mid = \mid J_p(u_i) \mid$$

so

$$\mid ML_p\backslash I \mid = \mid \{x,y\}A_p(r) \mid + \sum_{t=u_i+1}^{r-1} \mid \{y,z\}L_p(t)\backslash I \mid$$

$$= \mid A_p(r) \mid +1 + \sum_{t=u_i+1}^{r-1} (\mid L_p(t) \mid +1) = \mid L_p \mid +r - u_i$$

$$\mid MJ_p\backslash I \mid = \sum_{t=u_i}^{r-1} \mid \{y,z\}J_p(t)\backslash I \mid = \sum_{t=u_i+1}^{r-1} (\mid J_p(t) \mid +1) + \mid J_p(u_i) \mid$$

$$= \mid J_p \mid +r - u_i - 1 < \mid ML_p\backslash I \mid$$

a contradiction, so $\lambda \geq \mu$.

Next, we prove (12.6). Let $m > p+1 \geq q$, $F = (x^r) + I = (x^r) + R$, $U = S/F$, $V = S/R$, then F is a Macaulay-Lex ideal of S. Let $J_p \subseteq V_p$, $J_p \neq \emptyset$, E_p be the lex-segment of J_p in V_p, we need prove that $\mid ME_p\backslash R \mid \leq \mid MJ_p\backslash R \mid$. Let $B_p(t) = \{f \in V_p \mid deg_x(f) \geq t\}$, $g = \min(J_p)$.

Let $J_p \cap B_p(r) \neq \emptyset$. If $B_p(r)\backslash J_p \neq \emptyset$, then there exists $h \in B_p(r)\backslash J_p$ such that $f = yz^{-1}h$ or $y^{-1}zh \in J_p$, hence $yh = zf$ or $zh = yf \in M(J_p\backslash\{g\})$. Translate g to h until $J_p \subseteq B_p(r)$ or $B_p(r) \subseteq J_p$. If $B_p(r) \subseteq J_p$, then $B_p(r) \subseteq E_p$. Let $C_p = J_p\backslash B_p(r)$, $D_p = E_p\backslash B_p(r)$, then D_p is the lex-segment of C_p in $U_p = V_p\backslash B_p(r)$, hence $\mid MD_p\backslash F \mid \leq \mid MC_p\backslash F \mid$. Similar to (8.3) and (9.5), $\mid ME_p\backslash I \mid \leq \mid MJ_p\backslash I \mid$. If $J_p \subseteq B_p(r)$, then $E_p \subseteq B_p(r)$. For $i \geq \sigma$, since $r+v_\sigma \geq m > p+1$, $p-r+1 < v_\sigma \leq v_i$, so g_i does not divide x^ry^{p-r+1}, $R_{p+1}(r) = \emptyset$, $B_p(r) = x^rk[y,z]_{p-r}$, $x^{-r}E_p$ is the lex-segment of $x^{-r}J_p$ in $x^{-r}B_p(r) = k[y,z]_{p-r}$,

$$\mid ME_p\backslash I \mid = \mid \{y,z\}E_p \mid = \mid \{y,z\}(x^{-r}E_p) \mid$$

$$\leq \mid \{y,z\}(x^{-r}J_p) \mid = \mid \{y,z\}J_p \mid = \mid MJ_p \backslash I \mid$$

Let $J_p \cap B_p(r) = \emptyset$, then $J_p \subseteq W_p$. Let L_p be the lex-segment of J_p in W_p, $D = (g_i \mid \sigma \leq i \leq \alpha)$, then $D \subseteq I$, $\mid ML_p \backslash I \mid \leq \mid MJ_p \backslash I \mid$. Since $MJ_p \cap (x^{r+1}) = \emptyset$, $\mid MJ_p \backslash I \mid \leq \mid MJ_p \backslash D \mid = \mid MJ_p \backslash R \mid$. Since $\mid B_p(r) \mid =$
$\mid A_p(r) \mid$, if $\mid E_p \mid \leq \mid B_p(r) \mid$, then $E_p \subseteq B_p(r)$, $L_p \subseteq A_p(r)$. Similar to (12.7),

$$\mid ME_p \backslash R \mid = \mid \{y,z\}E_p \mid = \mid E_p \mid +1 = \mid L_p \mid +1 = \mid \{x,y\}L_p \mid$$

$$= \mid ML_p \backslash I \mid \leq \mid MJ_p \backslash I \mid \leq \mid MJ_p \backslash R \mid$$

Let $\mid E_p \mid > \mid B_p(r) \mid$, then $E_p \supseteq B_p(r)$. Let $C_p = E_p \backslash B_p(r)$, then $L_p = C_p \cup A_p(r)$. Since $\mid MB_p(r) \backslash R \mid = \mid B_p(r) \mid +1 = \mid A_p(r) \mid +1 = \mid MA_p(r) \backslash I \mid$, similar to (8.3),

$$\mid ME_p \backslash R \mid = \mid MB_p(r) \backslash R \mid + \mid MC_p \backslash F \mid = \mid MA_p(r) \backslash I \mid + \mid MC_p \backslash F \mid$$

$$= \mid ML_p \backslash I \mid \leq \mid MJ_p \backslash I \mid \leq \mid MJ_p \backslash R \mid$$

($\Leftarrow$): Let $J_p \subseteq W_p$, $J_p \neq \emptyset$, L_p be the lex-segment of J_p in W_p, we need prove that $\mid ML_p \backslash I \mid \leq \mid MJ_p \backslash I \mid$. Let $g = \min(J_p)$.

Let $p+1 \geq m$. If $A_p(r) \backslash J_p \neq \emptyset$ and $g \notin A_p(r)$, then there exists $h \in A_p(r) \backslash J_p$ such that $f = xy^{-1}h$ or $x^{-1}yh \in J_p \cup I$, hence $xh = yf$ or $yh = xf \in M(J_p \backslash \{g\}) \cup I$. Translate g to h until $J_p \subseteq A_p(r)$ or $A_p(r) \subseteq J_p$, similar to Proposition 10, $\mid ML_p \backslash I \mid \leq \mid MJ_p \backslash I \mid$.

If $p+1 < q$, let $F = (x^s, x^r z) + (g_i \mid 1 \leq i < \sigma)$, then $I_t = F_t (t \leq p+1)$. By Proposition 10, F is a Macaulay-Lex ideal of S. Similar to (8.1),
$\mid ML_p \backslash I \mid \leq \mid MJ_p \backslash I \mid$.

Let $q \leq p+1 < m$, then $p+1 < r + v_\sigma$. Let $i \geq \sigma$ be the smallest one such that $u_i + v_i + w_i \leq p+1$, by (12.5), $w_i = 0$. Since $p - v_i \leq p - v_\sigma < r-1$, for $p - v_i < t < r$, $p - t < v_i$, $x^t y^{p-t} \notin I$, $W_p(t) = S_p(t)$. Similarly, $I_{p+1}(t) = \emptyset (p - v_i + 1 < t < r)$.

If there exists t such that $p - v_i < t < r$ and $J_p(t) = \emptyset$, let $C_p = J_p \cap A_p(t)$, $K = (x^r z)$, D_p be the lex-segment of C_p in W_p, $Z_p = (J_p \backslash C_p) \cup D_p$, then K is a Macaulay-Lex ideal of S. Let $f = (e,c,d) \in MC_p \backslash K$, then $e > t$. For $0 \leq j < \sigma$, $deg(g_j) \geq m > p+1$, g_j does not divide f. For $\sigma \leq j < i$, $deg(g_j) > p+1$, g_j does not divide f. For $j \geq i$, $c = p+1-e-d \leq p-e+1 \leq p-t < v_i \leq v_j$, g_j does not divide f, so $f \in MC_p \backslash I$, $MC_p \backslash K \subseteq MC_p \backslash I$. Since $K \subseteq I$, $MC_p \backslash K = MC_p \backslash I$, so

$$\mid MD_p \backslash I \mid = \mid MD_p \backslash K \mid \leq \mid MC_p \backslash K \mid = \mid MC_p \backslash I \mid$$

$$\mid MZ_p \backslash I \mid = \mid M(J_p \backslash C_p) \backslash I \mid + \mid MD_p \backslash I \mid$$
$$\leq \mid M(J_p \backslash C_p) \backslash I \mid + \mid MC_p \backslash I \mid = \mid MJ_p \backslash I \mid$$

Replace J_p by Z_p, we can suppose that C_p is a lex-segment of W_p. If $C_p \neq \emptyset$, then $x^p \in J_p$. Translate g to $\max(W_p \backslash J_p)$ until $J_p \subseteq A_p(r)$ or $A_p(r) \subseteq J_p$, then $\mid ML_p\backslash I \mid \leq \mid MJ_p\backslash I \mid$. If $C_p = \emptyset$, let $V = S/R$, then $J_p \subseteq V_p$. Let E_p be the lex-segment of J_p in V_p, $G = (g_i \mid \sigma \leq i \leq \alpha)$, then $G \subseteq R$, $G \subseteq I$. Since $C_p = \emptyset$, $J_p \cap A_p(r-1) = \emptyset$, so $MJ_p \cap (x^{r+1}) = \emptyset$, $MJ_p \cap (x^r z) = \emptyset$, $\mid ME_p\backslash R \mid \leq \mid MJ_p\backslash R \mid = \mid MJ_p\backslash G \mid = \mid MJ_p\backslash I \mid$.

Let $B_p(t) = \{f \in V_p \mid deg_x(f) \geq t\}$. If $\mid L_p \mid \leq \mid A_p(r) \mid$, then $L_p \subseteq A_p(r)$, $E_p \subseteq B_p(r)$,

$$\mid ML_p\backslash I \mid = \mid L_p \mid +1 = \mid E_p \mid +1 = \mid ME_p\backslash R \mid \leq \mid MJ_p\backslash I \mid$$

If $\mid L_p \mid > \mid A_p(r) \mid$, let $D_p = L_p\backslash A_p(r)$, $F = (x^r) + I = (x^r) + R$, then $A_p(r) \subseteq L_p$, $B_p(r) \subseteq E_p$, $E_p = D_p \cup B_p(r)$. Similar to (8.3),

$$\mid ML_p\backslash I \mid = \mid MA_p(r)\backslash I \mid + \mid MD_p\backslash F \mid = \mid MB_p(r)\backslash R \mid + \mid MD_p\backslash F \mid$$

$$= \mid ME_p\backslash R \mid \leq \mid MJ_p\backslash R \mid = \mid MJ_p\backslash I \mid$$

Let $J_p(t) \neq \emptyset (p - v_i < t < r)$, $\mu = p - v_i + 1$. If $(A_p(\mu) \cap A_{p3}(1))\backslash J_p \neq \emptyset$, let $t \geq \mu$ be the greatest one such that $K_p = (W_p(t) \cap A_{p3}(1))\backslash J_p \neq \emptyset$. Since $J_p(t) \neq \emptyset$, there exists $h \in K_p$ such that $f_1 = yz^{-1}h$ or $y^{-1}zh \in J_p\backslash\{g\}$, hence $yh = zf_1$ or $zh = yf_1 \in M(J_p\backslash\{g\})$. Similar to (8.2), translate g to h until $A_p(\mu) \cap A_{p3}(1) \subseteq J_p$ or $J_p \cap A_{p3}(1)$ is a lex-segment of $A_p(\mu) \cap A_{p3}(1)$. If $J_p(\mu) = \emptyset$, similar to the above, $\mid ML_p\backslash I \mid \leq \mid MJ_p\backslash I \mid$.

Let $J_p(\mu) \neq \emptyset$, then $A_p(\mu+1) \cap A_{p3}(1) \subseteq J_p$, $J_p(\mu) \cap A_{p3}(1)$ is a lex-segment of $W_p(\mu) \cap A_{p3}(1)$. We reduce J_p such that $x^p \in J_p$, then $\mid ML_p\backslash I \mid \leq \mid MJ_p\backslash I \mid$ as above. Let $x^p \notin J_p$.

If there exists $\lambda > \mu$ such that $x^\lambda y^{p-\lambda} \in J_p$, let λ be the greatest one, then $\lambda < p$. Translate g to $x^{\lambda+1}y^{p-\lambda-1}$, $x^{\lambda+2}y^{p-\lambda-2}$, $\cdots$, x^p, then $x^p \in J_p$.

Let $x^t y^{p-t} \notin J_p(\mu < t \leq p)$. Since $J_p(\mu) \cap A_{p3}(1)$ is a lex-segment of $W_p(\mu) \cap A_{p3}(1)$, $f_2 = (\mu, v_i - 2, 1) \in J_p$ or $f_3 = (\mu, v_i - 1, 0) \in J_p$. If $\mid J_p(\mu) \mid = 1$, then $J_p(\mu) = \{f_j\}$, $g = \min(J_p) = f_j (j = 2, 3)$. If $g = f_2$, then $yg, zg \notin M(J_p\backslash\{g\}) \cup I$, $\mid MJ_p\backslash I \mid \geq \mid M(J_p\backslash\{g\})\backslash I \mid +2$. In fact, for $t \geq i$, $v_t \geq v_i$, g_t does not divide yg and zg. If $g = f_3$, then $xg, zg \notin M(J_p\backslash\{g\}) \cup I$. Similar to (8.2), translate g to x^p, then $x^p \in J_p$.

If $\mid J_p(\mu) \mid \geq 2$, then $f_2 \in J_p$. If $f_3 \notin J_p$, then $g \neq f_2$, $zf_3 = yf_2 \in M(J_p\backslash\{g\})$. Since $p - v_i + 1 \geq u_i$, $g_i \mid yf_3$, $yf_3 \in I$. Translate g to f_3, then $f_3 \in J_p$. Similarly, translate g to $xy^{-1}f_3$, $x^2y^{-2}f_3$, $\cdots$, x^p, then $x^p \in J_p$.

Example 1. $I = (x^{14}, x^9 z) + (x^{12}y^2, x^{10}y^4, x^5y^6, y^{11})$ and $J = (x^{14}, x^9z) + (x^{12}y^2, x^{10}y^4, x^5y^6, y^{10}z)$ are Macaulay-Lex ideals of $S = k[x, y, z]$ satisfying $q < m$.

For I and J, the condition (5.4) is the same, we need only consider I. $(x^{14}, x^{12}y^2, x^{10}y^4, x^9y^6)$ is a Macaulay-Lex ideal of $k[x, y]$, (12.2) holds. (x^{10}, x^5y^6, y^{11}) and (x^9, x^5y^6, y^{11}) are Macaulay-Lex ideals of $k[x, y]$ and S, (12.3), (12.6) hold. The other conditions hold also.

Proposition 13. Let $I = (x^s, x^r z) + (g_i \mid 1 \leq i \leq \alpha) + (h_i \mid 1 \leq i \leq \pi)$, $h_i = x^{a_i} y^{b_i}$, $g_i = x^{u_i} y^{v_i} z$, $s > r > u_1$, $u_i > u_{i+1}$, $1 \leq v_i < v_{i+1} (1 \leq i < \alpha)$, $s > a_i > a_{i+1}$, $1 \leq b_i < b_{i+1} (1 \leq i < \pi)$, $a_\pi \geq r$, $h_0 = x^s$, $a_0 = s$, $b_0 = 0$, then I is a Macaulay-Lex ideal of $S = k[x, y, z]$ if and only if I satisfies the following conditions

(13.1) Let $m = \min\{a_i + b_i \mid 0 \leq i \leq \pi\}$, $q = \min\{u_i + v_i + 1 \mid 1 \leq i \leq \alpha\}$, then $q \geq m$

(13.2) $(h_i \mid 0 \leq i \leq \pi)$ is a Macaulay-Lex ideal of $k[x, y]$

(13.3) $(x^r) + (g_i \mid 1 \leq i \leq \alpha)$ is a Macaulay-Lex ideal of S

Proof. ($\Rightarrow$): Similar to (12.5), if $q < m$, let $i \geq 1$ be the smallest one such that $p + 1 = q = u_i + v_i + 1$, $J_p = A_p(u_i) \cap A_{p3}(1)$, then $L_p = A_p(u_i + 1)$, $\mid ML_p \backslash I \mid > \mid MJ_p \backslash I \mid$, a contradiction, so $q \geq m$.

($\Leftarrow$): Let $J_p \subseteq W_p$, $J_p \neq \emptyset$, L_p be the lex-segment of J_p in W_p, we need prove that $\mid ML_p \backslash I \mid \leq \mid MJ_p \backslash I \mid$. If $p + 1 < q$, let $R = (x^s, x^r z) + (h_i \mid 1 \leq i \leq \pi)$, then R is a Macaulay-Lex ideal of S, $R_t = I_t (t \leq p + 1)$. Similar to (8.1), $\mid ML_p \backslash I \mid \leq \mid MJ_p \backslash I \mid$ holds. If $p + 1 \geq q$, then $p + 1 \geq m$. Translate $g = \min(J_p)$ into $A_p(r)$ until $J_p \subseteq A_p(r)$ or $A_p(r) \subseteq J_p$. Similar to Proposition 10, $\mid ML_p \backslash I \mid \leq \mid MJ_p \backslash I \mid$.

Similar to Proposition 13, we have

Proposition 14. Let $I = (x^s, x^r y) + (g_i \mid 1 \leq i \leq \alpha) + (h_i \mid 1 \leq i \leq \pi)$, $h_i = x^{a_i} z^{b_i}$, $g_i = x^{u_i} y^{v_i} z^{w_i}$, $s > r > u_1$, $u_i > u_{i+1}$, $1 \leq v_i \leq v_{i+1}$, $w_i \leq w_{i+1} \leq 1$, $2 \leq v_i + w_i < v_{i+1} + w_{i+1} (1 \leq i < \alpha)$, $s > a_i > a_{i+1}$, $1 \leq b_i < b_{i+1} (1 \leq i < \pi)$, $a_\pi \geq r$, $h_0 = x^s$, $a_0 = s$, $b_0 = 0$, then I is a Macaulay-Lex ideal of $S = k[x, y, z]$ if and only if I satisfies the following conditions

(14.1) Let $m = \min\{a_i + b_i \mid 0 \leq i \leq \pi\}$, $q = \min\{u_i + v_i + w_i \mid 1 \leq i \leq \alpha\}$, then $q \geq m$

(14.2) $(h_i \mid 0 \leq i \leq \pi)$ is a Macaulay-Lex ideal of $k[x, z]$

(14.3) $(x^r) + (g_i \mid 1 \leq i \leq \alpha)$ is a Macaulay-Lex ideal of S

In Proposition 14, let $\pi = 1 = b_1$, we have

Corollary 15. Let $I = (x^s, x^r z) + (g_i \mid 1 \leq i \leq \alpha)$, $g_i = x^{u_i} y^{v_i} z^{w_i}$, $s > r > u_1$, $v_1 = 1$, $w_1 = 0$, $u_i > u_{i+1}$, $v_i \leq v_{i+1}$, $w_i \leq w_{i+1} \leq 1$, $v_i + w_i < v_{i+1} + w_{i+1} (1 \leq i < \alpha)$, then I is a Macaulay-Lex ideal of $S = k[x, y, z]$ if and only if I satisfies the following conditions

(15.1) Let $q = \min\{u_i + v_i + w_i \mid 2 \leq i \leq \alpha\}$, then $q \geq r + 1$

(15.2) $(x^{u_1}) + (g_i \mid 2 \leq i \leq \alpha)$ is a Macaulay-Lex ideal of S

If $z(I) = r < s$ and $\Omega = \{x^{a_i} y^{b_i} z \mid 1 \leq i \leq m\} \cup \{x^{r-1} y^v\}$, then I is the ideal in the following Proposition 16.

Proposition 16. Let $I = (x^s, x^r z) + (h_i \mid 1 \leq i \leq \pi) + (g_i \mid 1 \leq i \leq \alpha)$, $h_i = x^{a_i} y^{b_i}$, $g_i = x^{u_i} y^{v_i} z$, $a_\pi = r - 1$, $s > r$, $s > a_i > a_{i+1}$, $1 \leq b_i < b_{i+1} (1 \leq$

$i < \pi$), $r > u_i > u_{i+1}$, $1 \le v_i < v_{i+1}(1 \le i < \alpha)$, $h_0 = x^s$, $a_0 = s$, $b_0 = 0$, $b_\pi > v_1$. If $u_1 = r-1$ and $\alpha \ge 2$, let $b_\pi > v_2$, then I is a Macaulay-Lex ideal of $S = k[x, y, z]$ if and only if I satisfies the following conditions

(16.1) Let $m = \min\{a_i + b_i \mid 0 \le i < \pi\}$, $q = \min\{u_i + v_i + 1 \mid 1 \le i \le \alpha\}$, then $q \ge m$

(16.2) $(h_i \mid 0 \le i < \pi) + (x^r y^{b_\pi})$ is a Macaulay-Lex ideal of $T = k[x, y]$, $r + b_\pi \ge m$

(16.3) If $\alpha \ge 2$, then $r + v_1 \le u_{\alpha-1} + v_\alpha$

(16.4) If $p + 1 < r + v_1$, then $E = (x^r) + (g_i \mid 1 \le i \le \alpha)$ satisfies the condition (ML3) for p

(16.5) $Y = (x^{r-1}) + (g_i \mid 1 \le i \le \alpha)$ is a Macaulay-Lex ideal of S

Proof. ($\Rightarrow$): (16.1) is similar to (12.5). (16.2) is similar to (12.2). Since $Y = (x^{r-1}) + I$, (16.5) holds. If $p + 1 < r + v_1$, then $p + 1 < a_\pi + b_\pi$. Let $F = (x^r) + I$, $V = S/E$, $U = S/F$, then F is a Macaulay-Lex ideal of S, $E_t = F_t$, $V_t = U_t(t \le p + 1)$. Similar to (8.1), (16.4) holds.

Next, we prove (16.3). By (5.2) and (16.5), if $u_1 = r - 1$, then $r - 1 + v_1 = u_1 + v_1 < u_1 + v_2 \le u_{\alpha-1} + v_\alpha$. Let $u_1 < r - 1$. If $r - 1 + v_1 \ge u_{\alpha-1} + v_\alpha$, then $r - 1 + v_1 \le u_1 + v_2 \le u_{\alpha-1} + v_\alpha \le r - 1 + v_1$, $r - 1 + v_1 = u_{i-1} + v_i(2 \le i \le \alpha)$. Let $p = u_1 + v_2 - 1 = r + v_1 - 2$, $f_2 = (u_1 - 1, v_2 - 1, 1)$, $J_p = A_p(r) \cup \{f_2\}$, $f_1 = x^{r-1}y^{p-r+1}$, then $f_2 \notin I$, $xf_2, yf_2 \in I$, $L_p = A_p(r) \cup \{f_1\}$. Since $p - r + 1 < v_1 < b_\pi$, h_π and $g_i(1 \le i \le \alpha)$ do not divide yf_1 and zf_1, hence $yf_1, zf_1 \notin I$,

$$| ML_p \backslash I | = | MA_p(r) \backslash I | + | \{yf_1, zf_1\} | = | MA_p(r) \backslash I | + 2$$

$$> | MA_p(r) \backslash I | + 1 = | MA_p(r) \backslash I | + | \{zf_2\} | \ge | MJ_p \backslash I |$$

a contradiction, so $r - 1 + v_1 < u_{\alpha-1} + v_\alpha$.

($\Leftarrow$): Let $J_p \subseteq W_p$, $J_p \ne \emptyset$, L_p be the lex-segment of J_p in W_p, we need prove that $| ML_p \backslash I | \le | MJ_p \backslash I |$. If $p+1 < q$, then $p+1 < u_1+v_1+1 \le a_\pi+b_\pi$. Let $R = (x^s, x^r z) + (h_i \mid 1 \le i < \pi)$, then R is a Macaulay-Lex ideal of S, $R_t = I_t(t \le p + 1)$. Similar to (8.1), $| ML_p \backslash I | \le | MJ_p \backslash I |$.

Let $p+1 \ge q$, then $p+1 \ge m$. Translate $g = \min(J_p)$ into $A_p(r)$ until $J_p \subseteq A_p(r)$ or $A_p(r) \subseteq J_p$. If $J_p \subseteq A_p(r)$, similar to Proposition 10, $| ML_p \backslash I | \le | MJ_p \backslash I |$. We can suppose that $A_p(r) \subseteq J_p$.

If $p + 1 < r + v_1$, similar to (8.3), let $B_p = J_p \backslash A_p(r)$, $C_p = L_p \backslash A_p(r)$, then $| MC_p \backslash E | \le | MB_p \backslash E |$, $| ML_p \backslash I | \le | MJ_p \backslash I |$.

Let $p+1 \ge r+v_1$, then $p-r+1 \ge v_1$, $(r-1, p-r+1, 1) \in I_{p+1}(r-1) \ne \emptyset$. Translate g to $h = \max(W_p(r-1) \backslash J_p)$ until $J_p = L_p \subseteq A_p(r-1)$ or $A_p(r-1) \subseteq J_p$, then $| ML_p \backslash I | \le | MJ_p \backslash I |$.

If $z(I) = r < s$ and $\Omega = \{x^{u_i}y^{v_i}z^{w_i} \mid 1 \le i \le m\} \cup \{x^{r-1}y^b z\}(w_1 = 0)$, then I is the ideal in the following Proposition 17.

Proposition 17. Let $I = (x^s, x^r z, x^{r-1}y^b z) + (g_i \mid 1 \le i \le \alpha)$, $g_i = x^{u_i}y^{v_i}z^{w_i}$, $s > r$, $s > u_i > u_{i+1}$, $1 \le v_i \le v_{i+1}$, $w_i \le w_{i+1} \le 1$, $1 \le v_i + w_i <$

$v_{i+1} + w_{i+1}(1 \leq i < \alpha)$, $g_0 = x^s$, $(u_0, v_0, w_0) = (s, 0, 0)$, $u_{\sigma-1} \geq r > u_\sigma$, $v_\sigma > b \geq 1$, $w_\sigma = 0$, then I is a Macaulay-Lex ideal of $S = k[x, y, z]$ if and only if I satisfies the following conditions

(17.1) Let $q = \min\{u_i + v_i + w_i \mid \sigma \leq i \leq \alpha\}$, then $q \geq r + 1$

(17.2) $R = (g_i \mid 0 \leq i < \sigma) + (x^r y^{v_\sigma})$ is a Macaulay-Lex ideal of $T = k[x, y]$.

(17.3) Let $m = \min\{u_i + v_i \mid 0 \leq i < \sigma\}$, then $r + b \geq m$

(17.4) If $w_\alpha = 1$, let $\lambda = \min\{u_i + v_i + 1 \mid \sigma \leq i \leq \alpha; w_i = 1\}$, $\mu = \min\{u_i + v_i \mid 0 \leq i < \alpha; w_i = 0\}$, then $\lambda \geq \mu$

(17.5) If $q \leq p + 1 < m$, then $F = (x^{r+1}) + (g_i \mid \sigma \leq i \leq \alpha)$ satisfies the condition (ML3) for p

(17.6) If $m \leq p + 1 < r + b$, then $Y = (x^r) + (g_i \mid \sigma \leq i \leq \alpha)$ satisfies the condition (ML3) for p

(17.7) $(x^{r-1}) + (g_i \mid \sigma \leq i \leq \alpha)$ is a Macaulay-Lex ideal of S

(17.8) If $\alpha > \sigma$, then $r + b < u_{\alpha-1} + v_\alpha + w_\alpha$

Proof. ($\Rightarrow$): We prove (17.3). If $p + 1 = r + b < m$, let $J_p = A_p(r-1) \cap A_{p3}(1)$, then $L_p = A_p(r)$. For $i \geq \sigma$, $p - r < b < v_\sigma \leq v_i$, $p - r + 1 < v_i$, so $x^{r-1}y^b z$ and $g_i(i \geq \sigma)$ do not divide $(r-1, p-r+1, 0)$ and $(r-1, p-r, 1)$, $I_p(r-1) = \emptyset$, $W_p(r-1) = S_p(r-1)$. Similar to the proof of (12.5), since $x^{r-1}y^b z \in I_{p+1}$,

$$\mid ML_p \backslash I \mid = \mid L_p \mid +1 = \mid J_p \mid +1 > \mid J_p \mid \geq \mid MJ_p \backslash I \mid$$

a contradiction, so $r + b \geq m$.

(17.8) is similar to (9.3) and (16.3). The other conditions are similar to Proposition 12.

($\Leftarrow$): Let $J_p \subseteq W_p$, $J_p \neq \emptyset$, L_p be the lex-segment of J_p in W_p, we need prove that $\mid ML_p \backslash I \mid \leq \mid MJ_p \backslash I \mid$.

If $p + 1 < q$ and $p + 1 < m$, let $E = (x^r z)$, then E is a Macaulay-Lex ideal of S, $E_t = I_t(t \leq p + 1)$. Similar to (8.1), $\mid ML_p \backslash I \mid \leq \mid MJ_p \backslash I \mid$.

If $q \leq p+1 < m$, by (17.5), similar to the case $q \leq p+1 < m$ of Proposition 12, $\mid ML_p \backslash I \mid \leq \mid MJ_p \backslash I \mid$.

If $p + 1 \geq m$, translate $g = \min(J_p)$ into $A_p(r)$ until $J_p \subseteq A_p(r)$ or $A_p(r) \subseteq J_p$. If $J_p \subseteq A_p(r)$, similar to Proposition 10, $\mid ML_p \backslash I \mid \leq \mid MJ_p \backslash I \mid$. We can suppose that $A_p(r) \subseteq J_p$.

If $m \leq p + 1 < r + b$, let $B_p = J_p \backslash A_p(r)$, $C_p = L_p \backslash A_p(r)$. By (17.6), $\mid MC_p \backslash Y \mid \leq \mid MB_p \backslash Y \mid$. Similar to (8.3), $\mid ML_p \backslash I \mid \leq \mid MJ_p \backslash I \mid$.

If $p + 1 \geq r + b$, then $(r-1, p-r+1, 1) \in I_{p+1}(r-1) \neq \emptyset$. Translate g to $\max(W_p(r-1) \backslash J_p)$ until $J_p = L_p \subseteq A_p(r-1)$ or $A_p(r-1) \subseteq J_p$. If $A_p(r-1) \subseteq J_p$, let $B_p = J_p \backslash A_p(r-1)$, $C_p = L_p \backslash A_p(r-1)$. Similar to (8.3), $\mid ML_p \backslash I \mid \leq \mid MJ_p \backslash I \mid$.

Example 2. $I = (x^{14}, x^9 z, x^8 y^6 z) + (x^{12}y^2, x^{10}y^4, x^5 y^7, y^{13})$ and $J = (x^{14}, x^9 z, x^8 y^6 z) + (x^{12}y^2, x^{10}y^4, x^5 y^7, y^{12}z)$ are Macaulay-Lex ideals of $S = k[x, y, z]$ satisfying $q < m$.

For I and J, the condition (5.4) is the same, we need only consider I. $(x^{14}, x^{12}y^2, x^{10}y^4, x^9y^7)$ is a Macaulay-Lex ideal of $k[x,y]$, (17.2) holds. (x^{10}, x^5y^7, y^{13}), (x^9, x^5y^7, y^{13}) and (x^8, x^5y^7, y^{13}) are Macaulay-Lex ideals of $k[x,y]$ and S, (17.5), (17.6), (17.7) hold. The other conditions hold also.

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