International Mathematical Forum, Vol. 8, 2013, no. 32, 1553 - 1559
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/imf.2013.37145

The Center of Coxeter Groups[1]

Zhaojin Zhou, Xigou Zhang and Yanni Zhou

College of Mathematics and information science
JXNU Nanchang, P.R. China, 330022
xyzhang71@sohu.com

Abstract. In this paper, we introduce the centers of Weyl groups. Let w_0^n be the longest element of Weyl group, then the center of Weyl group of type $B_n, C_n, D_{2n}, E_7, E_8, F_4, G_2, H_3, H_4$ is $\{1, w_0^n\}$; But in type A_n, D_{2n-1}, E_6, the center is 1.

Keywords: Weyl group; center; the longest element

Introduction

A Coxeter System is a pair (W, s) consisting of a group W and a set of generators S, subject only to relations of the form $(ss')^{m(s,s')} = 1$, where $m(s, s) = 1$, $m(s, s') = m(s', s) \geq 2$ for $s \neq s'$ in s. $W =< S_\alpha, \alpha \in \Delta >$, W is called a Coxeter group of type X_n. In this paper, $X_n = A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2, H_3, H_4$. Let $\Delta = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be the simple root system of Φ_n, $S = \{S_i | \alpha_i \in \Delta\}$, for $1 \leq i \leq n$, the simple reflection $S_i\alpha_i = -\alpha_i$, where $S_{\alpha_i} = S_i$. π be the positive system satisfing $\Delta \subseteq \pi$.Let w_0^n be the longest element of type X_n.

[1]This work was supported in part by the NSF of P.R. China(NO: 11261021).
The authors were also supported in part by the Science Foundation of Education Department of JX Province(NO:GJJ10396).

Let $c(X_n)$ be the center of X_n,then arbitray element $w \in c(X_n)$. We denote by $C(w_0^n)$, the conjugacy class of w_0^n in W of X_n.

Lemma 1(see [1]) Let Δ be a simple system, π be the corresponding positive system. w_0 be the longest element of W. Then w_0 is the unique element such that $w_0(\pi) = -\pi$.

Lemma 2 If $w \in c(X_n)$, then $w = 1$,or $w = w_0^n$, and when $w = w_0^n$, we have $w_0^n(\alpha) = -\alpha, \forall \alpha \in \pi$.

Proof : $\forall\ \alpha \in \pi$, we have $s_\alpha = w^{-1}s_\alpha w = s_{w(\alpha)}$ therefore we have $w(\alpha) = \alpha$, or $w(\alpha) = -\alpha$. when $w(\alpha) = \alpha$, $w = 1$. when $w(\alpha) = -\alpha$, we have $w(\pi) = -\pi$. so $w = w_0^n$, and w_0^n satisfy $w_0^n(\alpha) = -\alpha$, $\forall\ \alpha \in \pi$.

Lemma 3(see [2]) In the type of Weyl group of $E_7, E_8, F_4, G_2, H_3, H_4$, $|C(w_0^n)| = 1$; In the type of E_6, $|C(w_0^n)| > 1$.

1. For the Type A_{n-1}

Theorem 1 In the type A_{n-1}, $c(A_{n-1}) = 1$.

Proof : We consider the symmetric group S_n.

Let

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix}$$

be in the center of S_n.

Then $\forall\ \sigma \in S_n$,

$$\sigma\tau\sigma^{-1} = \begin{pmatrix} \sigma(1) & \sigma(2) & \cdots & \sigma(n) \\ \sigma(k_1) & \sigma(k_2) & \cdots & \sigma(k_n) \end{pmatrix} = \tau = \begin{pmatrix} 1 & 2 & \cdots & n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix}$$

If let $\sigma = (1, 2)$, then

$$\sigma\tau\sigma^{-1} = \begin{pmatrix} \sigma(1) & \sigma(2) & \cdots & \sigma(n) \\ \sigma(k_1) & \sigma(k_2) & \cdots & \sigma(k_n) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 3 & \cdots & n \\ \sigma(k_1) & \sigma(k_2) & \sigma(k_3) & \cdots & \sigma(k_n) \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 & \cdots & n \\ k_2 & k_1 & k_3 & \cdots & k_n \end{pmatrix}.$$

Hence

$$\begin{cases} \sigma(k_1) & = & k_2 \\ \sigma(k_2) & = & k_1 \end{cases}$$

so
$\begin{cases} k_1 &= 1 \\ k_2 &= 2 \end{cases}$ or $\begin{cases} k_1 &= 2 \\ k_2 &= 1 \end{cases}$
Now we can get

$$\begin{cases} k_1 &= 1 \\ k_2 &= 2 \end{cases}$$

otherwise

$$\begin{cases} k_1 &= 2 \\ k_2 &= 1 \end{cases}$$

then

$$\tau = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 1 & k_3 \cdots & k_n & \end{pmatrix}$$

let $\sigma = (2,3)$

$$\sigma\tau\sigma^{-1} = \begin{pmatrix} 1 & 3 & 2 & \cdots & n \\ 3 & 1 & \sigma(k_3) & \cdots & \sigma(k_n) \end{pmatrix} = \tau$$

but $k_1 = 2$
so

$$\begin{cases} k_1 &= 1 \\ k_2 &= 2 \end{cases}$$

and

$$\tau = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & k_3 & \cdots & k_n \end{pmatrix}$$

then let $\sigma = (2,3), (3,4), \cdots, (n-1,n)$, we have $k_i = i, (1 \leq i \leq n)$
Hence $\tau = 1$, $c(S_n) = 1$, and we have $c(A_{n-1}) = 1$.

2. For the Type $B_n, C_n (n \geq 2)$

Theorem 2 In the type B_n, C_n, the center is $\{1, w_0^n\}$.
Proof :We consider type C_n(since $B_n \cong C_n$).
Denote by $\varepsilon_1, \cdots, \varepsilon_n$ the standard basis of R^n.
For Δ take $\alpha_1 = \varepsilon_1, \alpha_2 = \varepsilon_1 - \varepsilon_2, \cdots, \alpha_{n-1} = \varepsilon_{n-2} - \varepsilon_{n-1}, \alpha_n = \varepsilon_{n-1} - \varepsilon_n$.
$w_0^n = S_n S_{n-1} \cdots S_2 S_1 S_2 \cdots S_n w_0^{n-1}$
Let $n = 2$, then $w_0^2 = S_1 S_2 S_1 S_2$.

$$w_0^2(\alpha_1) = S_1 S_2 S_1 S_2(\alpha_1) = S_1 S_2 S_1(\varepsilon_2) = S_1 S_2(\varepsilon_2) = -\varepsilon_1 = -\alpha_1$$

$$w_0^2(\alpha_2) = S_1S_2S_1S_2(\alpha_2) = S_1S_2S_1(\varepsilon_2 - \varepsilon_1) = S_1S_2(\varepsilon_2 + \varepsilon_1) = -\varepsilon_1 + \varepsilon_2 = -\alpha_2$$

So if $n = 2$, we have $w_0^2(\alpha_i) = -\alpha_i, i = 1, 2$
Suppose $n = k-1$, we have $w_0^{k-1}(\alpha_i) = -\alpha_i, 1 \le i \le k-1$
when $n = k$, we have $w_0^k = S_kS_{k-1}\cdots S_2S_1S_2\cdots S_{k-1}S_kw_0^{k-1}$
we have $w_0^k(\alpha_1) = S_kS_{k-1}\cdots S_2S_1S_2\cdots S_{k-1}S_k(-\alpha_1) = S_kS_{k-1}\cdots S_2(-\varepsilon_2) = -\varepsilon_1 = -\alpha_1$

$$\begin{aligned} w_0^k(\alpha_2) &= S_kS_{k-1}\cdots S_2S_1S_2\cdots S_{k-1}S_k(-\alpha_2) = S_kS_{k-1}\cdots S_2S_1S_2(\varepsilon_3 - \varepsilon_1) \\ &= S_kS_{k-1}\cdots S_2S_1(\varepsilon_3 - \varepsilon_2) = S_kS_{k-1}\cdots S_3(\varepsilon_3 - \varepsilon_1) \\ &= \varepsilon_2 - \varepsilon_1 = -\alpha_2 \end{aligned}$$

$\cdots$

$$\begin{aligned} w_0^k(\alpha_i) &= S_kS_{k-1}\cdots S_2S_1S_2\cdots S_{k-1}S_k(-\alpha_i) = S_kS_{k-1}\cdots S_2S_1S_2\cdots S_iS_{i+1}(\varepsilon_i - \varepsilon_{i-1}) \\ &= S_k\cdots S_1S_2\cdots S_i(\varepsilon_{i+1} - \varepsilon_{i-1}) = S_k\cdots S_1S_2\cdots S_{i-1}(\varepsilon_{i+1} - \varepsilon_i) \\ &= S_k\cdots S_{i+1}(\varepsilon_{i+1} - \varepsilon_{i-1}) = \varepsilon_i - \varepsilon_{i-1} = -\alpha_i \end{aligned}$$

$\cdots$

$$\begin{aligned} w_0^k(\alpha_k) &= S_k\cdots S_2S_1S_2\cdots S_kS_{k-1}\cdots S_2S_1S_2\cdots S_{k-1}(\varepsilon_{k-1} - \varepsilon_k) \\ &= S_k\cdots S_2S_1S_2\cdots S_kS_{k-1}\cdots S_2S_1S_2\cdots S_{k-2}(\varepsilon_{k-2} - \varepsilon_k) \\ &= S_k\cdots S_2S_1S_2\cdots S_kS_{k-1}\cdots S_2S_1(\varepsilon_1 - \varepsilon_k) \\ &= S_k\cdots S_2S_1S_2\cdots S_kS_{k-1}\cdots S_2(-\varepsilon_1 - \varepsilon_k) \\ &= S_k\cdots S_2S_1S_2\cdots S_kS_{k-1}(-\varepsilon_{k-2} - \varepsilon_k) = S_k\cdots S_2S_1S_2\cdots S_k(-\varepsilon_{k-1} - \varepsilon_k) \\ &= S_k\cdots S_2S_1S_2\cdots S_{k-2}(-\varepsilon_{k-2} - \varepsilon_k) = S_k\cdots S_2S_1(-\varepsilon_1 - \varepsilon_k) \\ &= S_k\cdots S_2(\varepsilon_1 - \varepsilon_k) = S_k(\varepsilon_{k-1} - \varepsilon_k) = \varepsilon_k - \varepsilon_{k-1} = -\alpha_k \end{aligned}$$

So $\forall\ \alpha_i \in \Delta$, $w_0^n(\alpha_i) = -\alpha_i$
Hence the center of B_n, C_n is $\{1, w_0^n\}$.

3. For the Type $D_n (n \ge 4)$

Theorem 3 In type $D_n, (n \ge 4)$, when n is even, $c(D_n) = \{1, w_0^n\}$; when n is odd, $c(D_n) = 1$.

Proof :Denote by $\varepsilon_1, \cdots, \varepsilon_n$ the standard basis of R^n.
For Δ take $\alpha_1 = \varepsilon_1 + \varepsilon_2, \alpha_2 = \varepsilon_1 - \varepsilon_2, \alpha_3 = \varepsilon_2 - \varepsilon_3, \cdots, \alpha_{n-1} = \varepsilon_{n-2} - \varepsilon_{n-1}, \alpha_n = \varepsilon_{n-1} - \varepsilon_n$
If $n = 4$, then $\alpha_1 = \varepsilon_1 + \varepsilon_2, \alpha_2 = \varepsilon_1 - \varepsilon_2, \alpha_3 = \varepsilon_2 - \varepsilon_3, \alpha_4 = \varepsilon_3 - \varepsilon_4$

$w_0^4 = S_4S_3S_2S_1S_4S_3S_2S_1S_4S_3S_2S_1$
So

$$\begin{aligned} w_0^4(\alpha_1) &= S_4S_3S_2S_1S_4S_3S_2S_1S_4S_3S_2(-\varepsilon_1-\varepsilon_2) = S_4S_3S_2S_1S_4S_3S_2S_1(-\varepsilon_1-\varepsilon_4) \\ &= S_4S_3S_2S_1S_4S_3S_2(\varepsilon_2-\varepsilon_4) = S_4S_3S_2S_1(\varepsilon_1-\varepsilon_3) \\ &= S_4S_3S_2(-\varepsilon_2-\varepsilon_3) = -\varepsilon_1-\varepsilon_2 = -\alpha_1 \end{aligned}$$

$$\begin{aligned} w_0^4(\alpha_2) &= S_4S_3S_2S_1S_4S_3S_2S_1S_4S_3S_2(\varepsilon_1-\varepsilon_2) = S_4S_3S_2S_1S_4S_3S_2S_1(\varepsilon_4-\varepsilon_1) \\ &= S_4S_3S_2S_1S_4S_3S_2(\varepsilon_2+\varepsilon_4) = S_4S_3S_2S_1(\varepsilon_1+\varepsilon_3) \\ &= S_4S_3S_2(\varepsilon_3-\varepsilon_2) = \varepsilon_2-\varepsilon_1 = -\alpha_2 \end{aligned}$$

$$\begin{aligned} w_0^4(\alpha_3) &= S_4S_3S_2S_1S_4S_3S_2S_1S_4S_3S_2(-\varepsilon_1-\varepsilon_3) = S_4S_3S_2S_1S_4S_3S_2S_1(-\varepsilon_2-\varepsilon_4) \\ &= S_4S_3S_2S_1(\varepsilon_4-\varepsilon_3) = S_4(-\varepsilon_2+\varepsilon_4) \\ &= \varepsilon_3-\varepsilon_2 = -\alpha_3 \end{aligned}$$

$$\begin{aligned} w_0^4(\alpha_4) &= S_4S_3S_2S_1S_4S_3S_2S_1S_4S_3(\varepsilon_3-\varepsilon_4) = S_4S_3S_2S_1S_4S_3S_2S_1(\varepsilon_2-\varepsilon_3) \\ &= S_4S_3S_2S_1S_4S_3S_2(-\varepsilon_1-\varepsilon_3) = S_4S_3S_2S_1(-\varepsilon_2-\varepsilon_4) \\ &= S_4S_3S_2(\varepsilon_1-\varepsilon_4) = \varepsilon_4-\varepsilon_3 = -\alpha_4 \end{aligned}$$

Therefore $c(D_4) = \{1, w_0^4\}$
If $n = 5$, then $\alpha_1 = \varepsilon_1+\varepsilon_2, \alpha_2 = \varepsilon_1-\varepsilon_2, \alpha_3 = \varepsilon_2-\varepsilon_3, \alpha_4 = \varepsilon_3-\varepsilon_4, \alpha_5 = \varepsilon_4-\varepsilon_5$

$$w_0^5 = S_5S_4S_3S_1S_2S_3S_4S_5S_4S_3S_2S_1S_4S_3S_2S_1S_4S_3S_2S_1$$

$$\begin{aligned} w_0^5(\alpha_1) &= S_5S_4S_3S_1S_2S_3S_4S_5(-\varepsilon_1-\varepsilon_2) = S_5S_4S_3S_1S_2(-\varepsilon_1-\varepsilon_3) \\ &= S_5S_4S_3S_1(-\varepsilon_2-\varepsilon_3) = S_5S_4S_3(\varepsilon_1-\varepsilon_3) \\ &= \varepsilon_1-\varepsilon_2 = \alpha_2 \end{aligned}$$

$$\begin{aligned} w_0^5(\alpha_2) &= S_5S_4S_3S_1S_2S_3S_4S_5(\varepsilon_2-\varepsilon_1) = S_5S_4S_3S_1S_2(\varepsilon_3-\varepsilon_1) \\ &= S_5S_4S_3S_1(\varepsilon_3-\varepsilon_2) = S_5S_4S_3(\varepsilon_1-\varepsilon_3) \\ &= \varepsilon_1+\varepsilon_3 = \varepsilon_1+\varepsilon_2 = \alpha_1 \end{aligned}$$

$$\begin{aligned} w_0^5(\alpha_3) &= S_5S_4S_3S_1S_2S_3S_4S_5(\varepsilon_3-\varepsilon_2) = S_5S_4S_3S_1S_2S_3(\varepsilon_4-\varepsilon_2) \\ &= S_5S_4S_3S_1(\varepsilon_4-\varepsilon_3) = S_5S_4(\varepsilon_4-\varepsilon_2) \\ &= \varepsilon_3-\varepsilon_2 = -\alpha_3 \end{aligned}$$

$$\begin{aligned} w_0^5(\alpha_4) &= S_5S_4S_3S_1S_2S_3S_4S_5(\varepsilon_4-\varepsilon_3) = S_5S_4S_3S_1S_2S_3S_4(\varepsilon_5-\varepsilon_3) \\ &= S_5S_4S_3S_1(\varepsilon_5-\varepsilon_4) = S_5(\varepsilon_5-\varepsilon_3) \\ &= \varepsilon_4-\varepsilon_3 = -\alpha_4 \end{aligned}$$

$$\begin{aligned} w_0^5(\alpha_5) &= S_5S_4S_3S_1S_2S_3S_4S_5S_4S_3S_2S_1S_4S_3S_2S_1S_4S_3S_2S_1(\varepsilon_3-\varepsilon_5) \\ &= S_5S_4S_3S_1S_2S_3S_4S_5S_4S_3S_2S_1(\varepsilon_2-\varepsilon_5) \\ &= S_5S_4S_3S_1S_2S_3S_4S_5S_4S_3S_2(-\varepsilon_1-\varepsilon_5) = S_5S_4S_3S_1S_2S_3S_4(-\varepsilon_4-\varepsilon_5) \\ &= S_5S_4S_3S_1(-\varepsilon_1-\varepsilon_5) = S_5S_4S_3(\varepsilon_2-\varepsilon_5) \\ &= \varepsilon_5-\varepsilon_4 = -\alpha_5 \end{aligned}$$

Suppose when n is odd we have $w_o^n(\alpha_1)=\alpha_2, w_o^n(\alpha_2)=\alpha_1, w_o^n(\alpha_i)=-\alpha_i, 3\le i\le n$. then in type D_{n+1}

$$\begin{aligned} w_0^{n+1}(\alpha_1) &= S_{n+1}S_n\cdots S_3S_1S_2\cdots S_nS_{n+1}(\varepsilon_1-\varepsilon_2) \\ &= S_{n+1}S_n\cdots S_3S_1S_2(\varepsilon_1-\varepsilon_3) \\ &= S_{n+1}S_n\cdots S_3(-\varepsilon_1-\varepsilon_3) = -\varepsilon_2-\varepsilon_1 = -\alpha_1 \end{aligned}$$

$$\begin{aligned} w_0^{n+1}(\alpha_2) &= S_{n+1}S_n\cdots S_3S_1S_2\cdots S_nS_{n+1}(\varepsilon_1+\varepsilon_2) \\ &= S_{n+1}S_n\cdots S_3S_1S_2(\varepsilon_1+\varepsilon_3) \\ &= S_{n+1}S_n\cdots S_3(\varepsilon_3-\varepsilon_1) = \varepsilon_2-\varepsilon_1 = -\alpha_2 \end{aligned}$$

$$\begin{aligned} w_0^{n+1}(\alpha_3) &= S_{n+1}S_n\cdots S_3S_1S_2\cdots S_nS_{n+1}(\varepsilon_3-\varepsilon_2) \\ &= S_{n+1}S_n\cdots S_3S_1S_2S_3(\varepsilon_4-\varepsilon_2) \\ &= S_{n+1}S_n\cdots S_3S_1(\varepsilon_4-\varepsilon_3) = S_{n+1}S_n\cdots S_4(\varepsilon_4-\varepsilon_2) \\ &= \varepsilon_3-\varepsilon_2 = -\alpha_3 \end{aligned}$$

$\cdots$

$$\begin{aligned} w_0^{n+1}(\alpha_i) &= S_{n+1}S_n\cdots S_3S_1S_2\cdots S_nS_{n+1}(\varepsilon_i-\varepsilon_{i-1}) \\ &= S_{n+1}S_n\cdots S_3S_1S_2\cdots S_i(\varepsilon_{i+1}-\varepsilon_{i-1}) \\ &= S_{n+1}S_n\cdots S_3S_1S_2\cdots S_{i-1}(\varepsilon_{i+1}-\varepsilon_i) \\ &= S_{n+1}S_n\cdots S_i(\varepsilon_{i+1}-\varepsilon_i) \\ &= S_{n+1}S_n\cdots S_{i+1}(\varepsilon_{i+1}-\varepsilon_{i-1}) = \varepsilon_i-\varepsilon_{i-1} = -\alpha_i \end{aligned}$$

$\cdots$

$$\begin{aligned} w_0^{n+1}(\alpha_n) &= S_{n+1}S_n\cdots S_3S_1S_2\cdots S_nS_{n+1}(\varepsilon_n-\varepsilon_{n-1}) \\ &= S_{n+1}S_n\cdots S_3S_1S_2\cdots S_{n-1}(\varepsilon_{n+1}-\varepsilon_n) \\ &= S_{n+1}S_n(\varepsilon_{n+1}-\varepsilon_n) = \varepsilon_n-\varepsilon_{n-1} = -\alpha_n \end{aligned}$$

$$\begin{aligned} w_0^{n+1}(\alpha_{n+1}) &= S_{n+1}S_n\cdots S_3S_1S_2\cdots S_nS_{n+1}S_nS_{n-1}\cdots S_3S_1S_2\cdots S_n(\varepsilon_n-\varepsilon_{n+1}) \\ &= S_{n+1}S_n\cdots S_3S_1S_2\cdots S_nS_{n+1}S_nS_{n-1}\cdots S_3S_1(\varepsilon_1-\varepsilon_{n+1}) \\ &= S_{n+1}S_n\cdots S_3S_1S_2\cdots S_nS_{n+1}S_nS_{n-1}\cdots S_3(-\varepsilon_2-\varepsilon_{n+1}) \\ &= S_{n+1}S_n\cdots S_3S_1S_2\cdots S_nS_{n+1}(-\varepsilon_n-\varepsilon_{n+1}) \\ &= S_{n+1}S_n\cdots S_3S_1(-\varepsilon_1-\varepsilon_{n+1}) \\ &= S_{n+1}S_n\cdots S_3(\varepsilon_2-\varepsilon_{n+1}) \\ &= S_{n+1}(\varepsilon_n-\varepsilon_{n+1}) = \varepsilon_{n+1}-\varepsilon_n = -\alpha_{n+1} \end{aligned}$$

Hence when n is even $c(D_n)=\{1,w_0^n\}$;when n is odd, $c(D_n)=1$.

4. For the Type $E_6, E_7, E_8, F_4, G_2, H_3, H_4$

Theorem 4 In type $E_7, E_8, F_4, G_2, H_3, H_4$, the center is $\{1, w_0^n\}$; In type E_6, the center is 1.

Proof : From lemma3, we can easily proof the theorem.

References

[1] James E Humphreys. The reflection groups and Coxeter groups [M]. Cambridge:Cambridge University Press, 1990.

[2] Meinolf Geck, pfeiffer. Characters of finite Coxeter groups and Iwahori-Hecke[M]. Oxford:Clarendon Press, 2000.

[3] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, Invent. Math. 53 (1979), 165C184.

[4] M. Dyer, Hecke algebras and reflections in Coxeter groups. Ph. D. thesis, University of Sydney, 1987.

[5] G.E. Murphy, The representations of Hecke algebras of type An, J. Algebra 173 (1995) 97-12.

[6] H. Wenzl, Hecke algebras of type A_n and subfactors, Invent. Math. 92 (1988) 349.

[7] Charles G. Cullen, Matrices and linear transformations [M]. Dover Publications, 1990.

[8] R. W. CARTER, Conjugacy Classes in the Weyl Group, 1972: 1-59.

Received: July 11, 2013