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On a Principal Ideal Domain that is not a Euclidean Domain

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Abstract

The ring $\mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$ is usually given as a first example of a principal ideal domain (PID) that is not a Euclidean domain. This paper gives an elementary and more direct proof that $\mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$ is indeed a PID.

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1 Introduction

In a course on abstract algebra, one proves that all Euclidean domains are principal ideal domains (PIDs). The ring $\mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$ is then usually given as a "simple" example of a PID that is not a Euclidean domain. However, details of this example are usually omitted. Some textbooks leave it as a series of exercises for the student. There have been efforts to simplify the proof that $\mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$ is indeed a PID but not a Euclidean domain, such as [6], [5] and, most recently, [2]. A comparative survey of the various papers can be found in [3].

For ease of notation, let $\omega = \frac{1+\sqrt{-19}}{2}$ henceforth.

It is straightforward to show that $\mathbb{Z}[\omega]$ in not Euclidean and this paper includes an existing proof for completeness. However, the proof that $\mathbb{Z}[\omega]$ is a PID is slightly more difficult. For example, the proofs in [6] and [3] leverage on a theorem due to Dedekind and Hasse, and the ensuing proof requires a breakdown into 5 cases, each corresponding to different elements of $\mathbb{Z}[\omega]$. The proof in [2] is a simplification, intended to make the material more accessible to mathematics students. However, it still requires a partitioning of $\mathbb{Z}[\omega]$ into 7 cases.

This paper provides an elementary and more direct proof that $\mathbb{Z}[\omega]$ is a PID. It is written with the same motivation as [2], utilising only introductory abstract algebra and the absolute value of a complex number, to improve access to comprehension. By partitioning $\mathbb{Z}[\omega]$ differently, the proof in this paper requires a breakdown into only 3 cases.

2 $\mathbb{Z}[\omega]$ is not a Euclidean Domain

This proof that $\mathbb{Z}[\omega]$ is not a Euclidean domain is similar to the proof in [2] and, as mentioned earlier, is included here for completeness.

Firstly, note that $\omega^2 = \omega - 5$. Thus, $\mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\}$. Also, as the minimal polynomial of ω over $\mathbb{Z}$ is $x^2 - x + 5$, which is Eisenstein and hence irreducible, $\mathbb{Z}[\omega]$ is an integral domain. For any element $\alpha \in \mathbb{Z}[\omega] \subset \mathbb{C}$, we have the usual absolute value $|\alpha| = \alpha\overline{\alpha}$, where $\overline{\alpha}$ denotes the usual complex conjugate of α. It is easy to see that for any $\alpha \in \mathbb{Z}[\omega]$, $\overline{\alpha} \in \mathbb{Z}[\omega]$ as well. We begin by proving some useful properties relating to the absolute values of elements in $\mathbb{Z}[\omega]$.

Lemma 2.1. *For $\alpha \in \mathbb{Z}[\omega] \setminus 0$, $|\alpha| \in \mathbb{N}$.*

Proof. As $\alpha = a + b\omega$ for some $a, b \in \mathbb{Z}$,

$$|\alpha| = [a + b(\tfrac{1+\sqrt{-19}}{2})][a + b(\tfrac{1-\sqrt{-19}}{2})] = a^2 + ab + 5b^2 \in \mathbb{Z}^{\geq 0}.$$

Since $\alpha \neq 0$, $|\alpha| \neq 0$. Thus, $|\alpha| \in \mathbb{N}$. □

Lemma 2.2. *For $\alpha \in \mathbb{Z}[\omega]$, the following statements are equivalent:*
(i) $\alpha = -1$ or 1.
(ii) α is a unit in $\mathbb{Z}[\omega]$.
(iii) $|\alpha| = 1$.

Proof. (i) $\Rightarrow$ (ii) is clear.

For (ii) $\Rightarrow$ (iii), if α is a unit in $\mathbb{Z}[\omega]$, then $\exists \beta \in \mathbb{Z}[\omega]$ such that $\alpha\beta = 1$. Then $1 = |\alpha\beta| = |\alpha||\beta|$. By Lemma 2.1, we must have $|\alpha| = |\beta| = 1$.

For (iii) $\Rightarrow$ (i), we write $\alpha = a + b\omega$ for some $a, b \in \mathbb{Z}$. Then $1 = |\alpha| = a^2 + ab + 5b^2 = (a + \frac{b}{2})^2 + \frac{19}{4}b^2$. As $a, b \in \mathbb{Z}$, we must have $b = 0$, which in turn implies that $a^2 = 1$. $\square$

Our proof that $\mathbb{Z}[\omega]$ is not Euclidean features some "special elements" of $\mathbb{Z}[\omega]$, namely $\pm 1, \pm 2$ and ± 3. Lemma 2.2 showed that ± 1 are the only units in $\mathbb{Z}[\omega]$. The following lemma shows that ± 2 and ± 3 are irreducible in $\mathbb{Z}[\omega]$. Recall that an element of a ring is *irreducible* if it satisfies the following properties:

(i) It is a nonzero non-unit in the ring; and

(ii) If it is written as a product of 2 elements of the ring, exactly 1 of them is a unit.

Lemma 2.3. ± 2 *and* ± 3 *are irreducible in* $\mathbb{Z}[\omega]$.

Proof. As ± 1 are units, it suffices to prove that 2 and 3 are irreducible.

2 is clearly a nonzero non-unit in $\mathbb{Z}[\omega]$, since $\frac{1}{2} \notin \mathbb{Z}[\omega]$. Suppose we write $2 = \alpha\beta$ for some $\alpha, \beta \in \mathbb{Z}[\omega]$. Then $4 = |2| = |\alpha||\beta|$. By Lemma 2.1, this implies that $(|\alpha|, |\beta|) = (1, 4), (2, 2)$ or $(4, 1)$. By Lemma 2.2, the first and the last cases would imply that either α or β is a unit respectively and, hence, 2 is irreducible.

For the case $(|\alpha|, |\beta|) = (2, 2)$, writing $\alpha = a + b\omega$ for some $a, b \in \mathbb{Z}$, we would get $2 = |\alpha| = a^2 + ab + 5b^2 = (a + \frac{b}{2})^2 + \frac{19}{4}b^2$. But then $a, b \in \mathbb{Z}$ means that $b = 0$, which in turn implies that $a^2 = 2$, a contradiction.

The proof that 3 is irreducible is similar. $\square$

Theorem 2.4. $\mathbb{Z}[\omega]$ *is not a Euclidean domain.*

Proof. Assume the contrary, i.e. that $\mathbb{Z}[\omega]$ is a Euclidean domain. Then there exists a Euclidean degree function $D : \mathbb{Z}[\omega] \setminus 0 \to \mathbb{N}$ satisfying the Euclidean Division Algorithm:

For $\alpha, \beta \in \mathbb{Z}[\omega]$ where $\beta \neq 0$, there exist $q, r \in \mathbb{Z}[\omega]$ such that $\alpha = \beta q + r$ and either $r = 0$ or $D(r) < D(\beta)$.

As the range of D is $\mathbb{N}$, we can choose $m \in \mathbb{Z}[\omega]$ such that $D(m)$ is as small as possible subject to m not being zero or a unit. Then let $q, r \in \mathbb{Z}[\omega]$ be the quotient and remainder, respectively, when we divide 2 by m in $\mathbb{Z}[\omega]$, i.e.

$$2 = mq + r, \quad \text{where } r = 0 \text{ or } D(r) < D(m).$$

$D(m)$ is already as small as possible subject to m being a nonzero non-unit. So either $r = 0$, or else $D(r) < D(m)$ implies that r is a unit in $\mathbb{Z}[\omega]$, i.e. $r = -1$ or 1 (by Lemma 2.2).

If $r = 0$, then m divides 2. Since m is not a unit and 2 is irreducible in $\mathbb{Z}[\omega]$ (by Lemma 2.3), this means that $m = -2$ or 2. (Again, we have used the fact that the only units in $\mathbb{Z}[\omega]$ are -1 and 1.)

If $r = -1$, then m divides 3. By a similar line of reasoning as in the case above, $m = -3$ or 3.

If $r = 1$, then m divides 1, which is a contradiction since m is not a unit by assumption.

Thus, we have shown that the possible choices for m (i.e. the nonzero non-unit elements of $\mathbb{Z}[\omega]$ with minimal degree D) are ± 2 and ± 3.

Next, we divide ω by m in $\mathbb{Z}[\omega]$, getting

$$\omega = mq' + r', \quad \text{for some } q', r' \in \mathbb{Z}[\omega] \text{ where } r' = 0 \text{ or } D(r') < D(m).$$

By the same argument as above, this implies that $r' = -1, 0$ or 1.

If $r' = -1$, then m divides $1+\omega$ in $\mathbb{Z}[\omega]$. But as $m \in \{\pm 2, \pm 3\}$, $\frac{1}{m}(1+\omega) \notin \mathbb{Z}[\omega]$, a contradiction.

If $r' = 0$, then m divides ω in $\mathbb{Z}[\omega]$. But as $m \in \{\pm 2, \pm 3\}$, $\frac{1}{m}(\omega) \notin \mathbb{Z}[\omega]$, a contradiction.

If $r' = 1$, then m divides $-1+\omega$ in $\mathbb{Z}[\omega]$. But as $m \in \{\pm 2, \pm 3\}$, $\frac{1}{m}(-1+\omega) \notin \mathbb{Z}[\omega]$, a contradiction. $\square$

3 $\mathbb{Z}[\omega]$ is a Principal Ideal Domain

This proof is based on a combination of ideas from [1] and [7]. Importantly, it hinges on the absolute values of elements in $\mathbb{Z}[\omega]$ and, thus, uses Lemma 2.1 from the previous section.

Theorem 3.1. *$\mathbb{Z}[\omega]$ is a principal ideal domain.*

Proof. Let I be any nonzero ideal in $\mathbb{Z}[\omega]$. As Lemma 2.1 showed that the absolute values of nonzero elements in $\mathbb{Z}[\omega]$ are natural numbers, we can pick a nonzero $\beta \in I$ such that $|\beta|$ is as small as possible among the nonzero elements of I. We seek to show that $I = (\beta)$, i.e. I is a principal ideal generated by β.

Assume the contrary. Then there exists a nonzero $\alpha \in I \setminus (\beta)$. Consider $\frac{\alpha}{\beta} \in \mathbb{C}$. As $\omega = \frac{1}{2} + \frac{\sqrt{19}}{2} i \in \mathbb{C}$, we can pick $m \in \mathbb{Z}$ such that

$$-\tfrac{\sqrt{19}}{4} < Im(\tfrac{\alpha}{\beta} + m\omega) \leq \tfrac{\sqrt{19}}{4}$$

where Im refers to the imaginary part of a complex number. We now split up the argument into 2 cases, depending on the value of $Im(\frac{\alpha}{\beta} + m\omega)$.

<u>Case 1</u>. $-\frac{\sqrt{3}}{2} < Im(\frac{\alpha}{\beta} + m\omega) < \frac{\sqrt{3}}{2}$

In this more straightforward case, we can pick $n \in \mathbb{Z}$ such that

$$-\tfrac{1}{2} < Re(\tfrac{\alpha}{\beta} + m\omega + n) \leq \tfrac{1}{2}$$

where Re refers to the real part of a complex number. Since $Im(\frac{\alpha}{\beta} + m\omega + n) = Im(\frac{\alpha}{\beta} + m\omega)$, we also have

$$-\tfrac{\sqrt{3}}{2} < Im(\tfrac{\alpha}{\beta} + m\omega + n) < \tfrac{\sqrt{3}}{2}.$$

Thus, $|\frac{\alpha}{\beta} + m\omega + n| < (\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2 = 1$, and $|\alpha + (m\omega + n)\beta| = |\frac{\alpha}{\beta} + m\omega + n||\beta| < |\beta|$.

But as $\alpha, \beta \in I$ and $m\omega + n \in \mathbb{Z}[\omega]$, it follows that $\alpha + (m\omega + n)\beta \in I$. Since $|\beta|$ is as small as possible among the absolute values of nonzero elements in I, $|\alpha + (m\omega + n)\beta| < |\beta|$ implies that $\alpha + (m\omega + n)\beta = 0$. Thus, $\alpha \in (\beta)$, which contradicts our assumption.

<u>Case 2</u>. Either $-\frac{\sqrt{19}}{4} < Im(\frac{\alpha}{\beta} + m\omega) \leq -\frac{\sqrt{3}}{2}$, or $\frac{\sqrt{3}}{2} \leq Im(\frac{\alpha}{\beta} + m\omega) \leq \frac{\sqrt{19}}{4}$

If $-\frac{\sqrt{19}}{4} < Im(\frac{\alpha}{\beta} + m\omega) \leq -\frac{\sqrt{3}}{2}$, then let $\alpha' = -\alpha - m\omega\beta$.
If $\frac{\sqrt{3}}{2} \leq Im(\frac{\alpha}{\beta} + m\omega) \leq \frac{\sqrt{19}}{4}$, then let $\alpha' = \alpha + m\omega\beta$.

In both instances, since $\alpha, \beta \in I$ and $m, \omega \in \mathbb{Z}[\omega]$, we see that $\alpha' \in I$. But if $\alpha' \in (\beta)$, then $\alpha = \mp(\alpha' - m\omega\beta) \in (\beta)$ as well, which contradicts our assumption that $\alpha \notin (\beta)$. Thus, in both instances, we have found an element $\alpha' \in I \setminus (\beta)$ such that

$$\tfrac{\sqrt{3}}{2} \leq Im(\tfrac{\alpha'}{\beta}) \leq \tfrac{\sqrt{19}}{4}.$$

Now, as in <u>Case 1</u>, we can find $n \in \mathbb{Z}$ such that

$$-\tfrac{1}{2} < Re(\tfrac{\alpha'}{\beta} + n) \leq \tfrac{1}{2}.$$

Let $\alpha'' = \alpha' + n\beta \in I$. Note that $Im(\frac{\alpha''}{\beta}) = Im(\frac{\alpha'}{\beta})$. As before, if $\alpha'' \in (\beta)$, then $\alpha' = \alpha'' - n\beta \in (\beta)$ as well, which is a contradiction. Thus, we have found an element $\alpha'' \in I \setminus (\beta)$ such that

$$\tfrac{\sqrt{3}}{2} \leq Im(\tfrac{\alpha''}{\beta}) \leq \tfrac{\sqrt{19}}{4}, \text{ and } -\tfrac{1}{2} < Re(\tfrac{\alpha''}{\beta}) \leq \tfrac{1}{2}.$$

To finish the proof, we consider the element $\frac{2\alpha''}{\beta} - \omega \in \mathbb{C}$, which will give us the desired contradictions via 2 subcases. Since $\omega = \frac{1}{2} + \frac{\sqrt{19}}{2}i$, we get that

$$-\tfrac{3}{2} < Re(\tfrac{2\alpha''}{\beta} - \omega) \leq \tfrac{1}{2}.$$

Noting that $\sqrt{19} < \sqrt{27} = 3\sqrt{3}$, we get $\sqrt{3} - \frac{\sqrt{19}}{2} > \sqrt{3} - \frac{3\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$. Thus,

$$-\tfrac{\sqrt{3}}{2} < \sqrt{3} - \tfrac{\sqrt{19}}{2} \leq Im(\tfrac{2\alpha''}{\beta} - \omega) \leq 0.$$

Case 2(a). $-\frac{1}{2} < Re(\frac{2\alpha''}{\beta} - \omega) \leq \frac{1}{2}$

In this sub-case, since $|\frac{2\alpha''}{\beta} - \omega| < (\frac{1}{2})^2 + (-\frac{\sqrt{3}}{2})^2 = 1$, we see that $|2\alpha'' - \omega\beta| = |\frac{2\alpha''}{\beta} - \omega||\beta| < |\beta|$. Since $\alpha'', \beta \in I$, it follows that $2\alpha'' - \omega\beta \in I$ as well. But as $|\beta|$ is as small as possible among the absolute values of nonzero elements in I, $|2\alpha'' - \omega\beta| < |\beta|$ implies that $2\alpha'' - \omega\beta = 0$. This means that $\frac{\omega\beta}{2} = \alpha'' \in I$.

Now as $\overline{\omega} \in \mathbb{Z}[\omega]$ and $\overline{\omega}\omega = 5$, we have $\frac{5}{2}\beta = \overline{\omega}(\frac{\omega\beta}{2}) \in I$. And since $\beta \in I$, we see that $\frac{1}{2}\beta = \frac{5}{2}\beta - 2\beta \in I$ as well. But then $0 < |\frac{1}{2}\beta| = \frac{1}{4}|\beta| < |\beta|$ contradicts the minimality of $|\beta|$ among the absolute values of nonzero elements in I, which completes the proof of this sub-case.

Case 2(b). $-\frac{3}{2} < Re(\frac{2\alpha''}{\beta} - \omega) \leq -\frac{1}{2}$

In this sub-case, we "shift by 1" to get a proof similar to Case 2(a), i.e. we consider $\frac{2\alpha''}{\beta} - \omega + 1 \in \mathbb{C}$. Clearly,

$$-\tfrac{1}{2} < Re(\tfrac{2\alpha''}{\beta} - \omega + 1) \leq \tfrac{1}{2}, \text{ and } -\tfrac{\sqrt{3}}{2} < Im(\tfrac{2\alpha''}{\beta} - \omega + 1) \leq 0$$

since $Im(\frac{2\alpha''}{\beta} - \omega + 1) = Im(\frac{2\alpha''}{\beta} - \omega)$.

Thus, $|\frac{2\alpha''}{\beta} - \omega + 1| < (\frac{1}{2})^2 + (-\frac{\sqrt{3}}{2})^2 = 1$, and we see that $|2\alpha'' - \omega\beta + \beta| = |\frac{2\alpha''}{\beta} - \omega + 1||\beta| < |\beta|$. Since $\alpha'', \beta \in I$, it follows that $2\alpha'' - \omega\beta + \beta \in I$ as well. But as $|\beta|$ is as small as possible among the absolute values of nonzero elements in I, $|2\alpha'' - \omega\beta + \beta| < |\beta|$ implies that $2\alpha'' - \omega\beta + \beta = 0$. This means that $\frac{\omega-1}{2}\beta = \alpha'' \in I$.

Now as $\overline{\omega - 1} \in \mathbb{Z}[\omega]$ and $(\overline{\omega - 1})(\omega - 1) = 5$, we have $\frac{5}{2}\beta = (\overline{\omega - 1})(\frac{\omega-1}{2}\beta) \in I$. By an argument identical to that in Case 2(a), $\frac{1}{2}\beta \in I$ as well, contradicting the minimality of $|\beta|$ among the absolute values of nonzero elements in I and completing the proof. $\square$

4 Concluding Remarks

The ring $\mathbb{Z}[\omega]$ is an example of a quadratic integer ring. In general, for a square-free integer D, let

$$\theta = \begin{cases} \sqrt{D} & \text{if } D \equiv 2, 3 \pmod 4 \\ \frac{1+\sqrt{D}}{2} & \text{if } D \equiv 1 \pmod 4 \end{cases}$$

Then, $\mathbb{Z}[\theta]$ is a quadratic integer ring (the ring of integers in the quadratic number field, $\mathbb{Q}(\sqrt{D})$).

It is known that $\mathbb{Z}[\theta]$ is a PID but not a Euclidean domain exactly when $D = -19, -43, -67$ or -163 (see [3], [4] and [5]). This paper dealt with the case $D = -19$. Perhaps a possible next step would be to find a unifying proof (for all 4 cases) that is equally accessible to students in mathematics.

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