International Mathematical Forum, Vol. 8, 2013, no. 32, 1597 - 1603
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/imf.2013.38162

On Orthogonal Generalized Higher α-Derivation of Γ-Ring M

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Abstract

Let M be a Γ-ring and $\alpha_n = (\alpha_i)_{i \in N}$ be a family of an endomorphism mappings of M. In this paper we will present and study the concepts of generalized higher α-derivation, Jordan generalized higher α-derivation and Jordan triple generalized higher α-derivation on Γ-ring.

The main results are present and study every two generalized higher α-derivations (D_n,d_n) and (G_n,g_n) are orthogonal.

Mathematics Subject Classification: 16W25, 16Y30, 16A72

Keywords: Derivation, α-derivation, higher α-derivation, generalized higher α-derivation, orthogonal generalized higher α-derivation

1. Introduction

Let R is a ring and M be a Γ-ring, M is called prime if $a\Gamma M\Gamma b = 0$ implies that $a = 0$ or $b = 0$ and it's said to be semiprime if $a\Gamma M\Gamma a = 0$ implies $a = 0$, for all $a, b \in$ M. Also M is called 2-torsion free if $2a = 0$, $a \in$ M implies $a = 0$.

An additive mapping d:R $\longrightarrow$ R is called derivation if d(ab) = d(a)b + ad(b), and be a Jordan derivation if d(a^2) = d(a)a + ad(a), for all a, $b \in$ R. F.J.Jing in [6] and M.Sapanci in [8] are defined a derivation and Jordan derivation on Γ-ring (resp.) as follows: for all a, $b \in$ M and $\gamma \in \Gamma$, then the additive mapping d: M $\longrightarrow$ M is called a derivation if d($a\gamma b$) = d(a)γb + $a\gamma$d(b), and d is called Jordan derivation if d($a\gamma a$) = d(a)γa + $a\gamma$d(a).
Y.Ceven and M.Ozturk in [4] are defined a generalized derivation on Γ-ring as follows: for all a, $b \in$ M and $\gamma \in \Gamma$, then D($a\gamma b$) = D(a)γb + $a\gamma$d(b).
A.H.Majeed and S.M.Salih in [7] are introduced the definition of higher derivation on Γ-ring by form: Let d = $(d_i)_{i \in N}$ be a family of additive mapping of Γ-ring M such that $d_0 = id_M$. Then d is called a higher derivation of M if for every n $\in$ N, a, $b \in$ M and $\gamma \in \Gamma$,then $d_n(a\gamma b) = \sum_{i+j=n} d_i(a)\gamma d_j(b)$, and defined the Jordan higher derivation as follows: $d_n(a\gamma a) = \sum_{i+j=n} d_i(a)\gamma d_j(a)$; while they defined the Jordan triple higher derivation by $d_n(a\gamma b\beta a) = \sum_{i+j+l=n} d_i(a)\gamma d_j(b)\beta d_l(a)$, for all a, $b \in$ M and $\gamma, \beta \in \Gamma$.
J.C.Chang in [3] is defined α-derivation on ring R by: an additive mapping d: R $\longrightarrow$ R is called α-derivation if for all a, $b \in$ R, then d(ab) = d(a)α(b) + ad(b). And we extend these definition on Γ-ring as follows: d($a\gamma b$) = d(a)$\gamma\alpha$(b) + $a\gamma$d(b), for all a, $b \in$ M and $\gamma \in \Gamma$.
Also we present and study concepts of higher α-derivation, Jordan higher α-derivation and Jordan triple higher α-derivation on M as follows:
$d_n(a\gamma b) = \sum_{i+j=n} d_i(a)\gamma\alpha_i(d_j(b))$, for all a, $b \in$ M and $\gamma \in \Gamma$.
A.K.Faraj in [5] introduced the concepts of higher homomorphism, Jordan higher homomorphism and Jordan triple higher homomorphism. M.Bresar and J.Vukman in [2] are introduced the notion of orthogonality for a pair d, g of derivations on a semiprime ring R. Also we present and study the concept of orthogonal generalized α-derivation on M.

In this paper we will extend these results to orthogonal generalized higher α-derivation on M and obtain some results parallel to these earlier obtained by N.Argac, A.Nakajima and E.Albas in [1].

Throughout this paper, R is a ring and M be a 2-torsion semiprime Γ-ring.

2. Generalized Higher α-Derivation on Γ-Ring M

In this section we present and study the definitions of generalized higher α-derivation, Jordan generalized higher α-derivation and Jordan triple generalized higher α-derivation on Γ-ring M.

Definition (2.1): Let M be a Γ-ring and $\alpha = (\alpha_i)_{i \in N}$ be a family of endomorphism mappings of M, such that $\alpha_0 = id_M$ and let $D_n = (D_i)_{i \in N}$ be a family of an additive mappings of M, such that $D_0 = id_M$. Then D_n is called generalized higher α-derivation if there exist a higher α-derivation $d = (d_i)_{i \in N}$ of M, such that for every $a, b \in M$ and $\gamma \in \Gamma$ then

$$D_n(a\gamma b) = \sum_{i+j=n} D_i(a)\gamma\alpha_i(d_j(b)), n \in N$$

…(1)

And D_n is a Jordan generalized higher α-derivation if

$$D_n(a\gamma a) = \sum_{i+j=n} D_i(a)\gamma\alpha_i(d_j(a)), n \in N$$

…(2)

D_n is called a Jordan triple generalized higher α-derivation if $n \in N$, we have

$$D_n(a\gamma b\beta a) = \sum_{i+j+l=n} D_i(a)\gamma\alpha_i(d_j(b))\beta\alpha_i(\alpha_j(d_l(a))) \quad \text{for all } a, b \in M \text{ and } \gamma, \beta \in \Gamma$$

…(3)

Example (2.2): Let $d = (d_i)_{i \in N}$ be a family of higher α-derivation of Γ-ring M. Put $S = M \oplus M$, $\alpha' = (\alpha'_i)_{i \in N}$ be a family of endomorphism mappings of S defined by

$\alpha'_n((x,y)) = (\alpha_n(x), \alpha_n(y))$, for all $(x,y) \in S$.

And we define $d' = (d'_i)_{i \in N}$ be a family of additive mappings on S by

$d'_n((x,y)) = (d_n(x),0)$ for all $(x,y) \in S$, then d' is higher α-derivation of S. Moreover if (D_n,d_n) is generalized higher α-derivation of M and we defined

$D'_n((x,y)) = (D_n(x),0)$, for all $(x,y) \in S$.

Then (D'_n, d'_n) is generalized higher α'-derivation of S.

Now, we introduce some properties of generalized higher α-derivation on Γ-ring .

Lemma (2.3): Let $\alpha = (\alpha_i)_{i \in N}$ be a family of endomorphism mappings of Γ-ring M, and $D_n = (D_i)_{i \in N}$ be a Jordan generalized higher α-derivation on M. Then for every $a, b, c \in M$, $\gamma, \beta \in \Gamma$ and $n \in N$ the following statements hold:

(i) $D_n(a\gamma b + b\gamma a) = \sum_{i+j=n} D_i(a)\gamma\alpha_i(d_j(b)) + D_i(b)\gamma\alpha_i(d_j(a))$

(ii)

$$D_n(a\gamma b\beta a + a\beta b\gamma a) = \sum_{i+j+l=n} D_i(a)\gamma\alpha_i(d_j(b))\beta\alpha_i(\alpha_j(d_l(a))) + D_i(a)\beta\alpha_i(d_j(b))\gamma\alpha_i(\alpha_j(d_l(a)))$$

(iii) In particular if M be a 2-torsion free commutative Γ-ring then

$$D_n(a\gamma b\gamma a) = \sum_{i+j+l=n} D_i(a)\gamma\alpha_i(d_j(b))\gamma\alpha_i(\alpha_j(d_l(a)))$$

(iv)

$$D_n(a\gamma b\beta c + c\gamma b\beta a) = \sum_{i+j+l=n} D_i(a)\gamma\alpha_i(d_j(b))\beta\alpha_i(\alpha_j(d_l(c))) +$$

$$D_i(c)\gamma\alpha_i(d_j(b))\beta\alpha_i(\alpha_j(d_l(a)))$$

(v)

$$D_n(a\gamma b\gamma c + c\gamma b\gamma a) = \sum_{i+j+l=n} D_i(a)\gamma\alpha_i(d_j(b))\gamma\alpha_i(\alpha_j(d_l(c))) +$$

$$D_i(c)\gamma\alpha_i(d_j(b))\gamma\alpha_i(\alpha_j(d_l(a)))$$

Proof : **(i)** is obtained by computing $D_n((a + b)\ \gamma\ (b + a))$ and **(ii)** is also obtained by replacing $a\beta b + b\beta a$ for b in (i), moreover **(iii)** can be obtained by replacing β with γ in (ii). If we replace $a + c$ for a in definition 2.1 (3) we can get (iv). While if replacing β by γ in (iv) we obtained (v).

3. Main Results

In this section, we introduce the definition of orthogonal generalized higher α-derivation on Γ-ring M. Also study the concepts of orthogonal higher α-derivations on M.

We begin with the following definition:

Definition (3.1): Let M be Γ-ring and $\alpha = (\alpha_i)_{i\in N}$ be a family of endomorphism mappingsof M, $D_n = (D_i)_{i\in N}$ and $G_n = (G_i)_{i\in N}$ are generalized higher α-derivation of M. Then D_n and G_n are called orthogonal if for every $n \in N$ and $x, y \in M$ we have

$D_n(x)\Gamma\alpha_n(M)\Gamma G_n(y) = (0) = G_n(y)\Gamma\alpha_n(M)\Gamma D_n(x)$, where $\sum_{i=1}^{n} D_i(x)\Gamma\alpha_i(M)\Gamma G_i(y) = 0$.

Example (3.2): Let M be a Γ-ring, (D'_n, d'_n) and (G'_n, g'_n) are generalized higher α'-derivations of S as in example (2.2). Then D'_n and G'_n are orthogonal generalized higher α'-derivation on S.

Lemma (3.3)([9], lemma 3): Let M be a 2-torsion free semiprime Γ-ring and a, b the elements of M. Then the following conditions are equivalent:

(i) $a\gamma M\beta b = (0)$.

(ii) $b\gamma M\beta a = (0)$.

(iii) $a\gamma M\beta b + b\gamma M\beta a = (0)$, for all $\gamma, \beta \in \Gamma$.

If one of these conditions are fulfilled then $a\gamma b = b\gamma a = 0$, for all $\gamma \in \Gamma$.

Lemma (3.4): Let $\alpha = (\alpha_i)_{i \in N}$ be a family of endomorphism mappings of M. If $D_n = (D_i)_{i \in N}$ and $G_n = (G_i)_{i \in N}$ are orthogonal α-derivations acts as a generalized higher homomorphism of Γ-ring M. Then the following relations hold for all x, y ∈ M and γ ∈ Γ, n ∈ N

(i) $D_n(x)\gamma G_n(y) = G_n(x)\gamma D_n(y) = (0)$, hence $D_n(x)\gamma G_n(y) + G_n(x)\gamma D_n(y) = (0)$.

In the particular if α_n are commuting mappings then

(ii) d_n and G_n are orthogonal higher derivations and $d_n(\alpha_n(x))\gamma G_n(y) = G_n(y)\gamma d_n(\alpha_n(x)) = 0$.

(iii) g_n and D_n are orthogonal higher derivations and $g_n(\alpha_n(x))\gamma D_n(y) = D_n(y)\gamma g_n(\alpha_n(x)) = 0$.

(iv) g_n and d_n are orthogonal higher derivation.

(v) $d_nG_n = G_nd_n = (0)$ and $g_nD_n = D_ng_n = (0)$.

(vi) $D_nG_n = G_nD_n = (0)$.

Proof : **(i)** By the hypothesis we have $D_n(x)\Gamma\alpha_n(z)\Gamma G_n(y) = 0$ for all x, y, z ∈ M. Hence by lemma (3.3) we get $\sum_{i=1}^{n} D_i(x)\gamma G_i(y) = (0) = \sum_{i=1}^{n} G_i(x)\gamma D_i(y)$, for all x, y ∈ M and γ ∈ Γ.

(ii) By (i) we have $D_n(x)\gamma G_n(y) = (0)$ and by the hypothesis we get $\sum_{i=1}^{n} D_i(m\beta x)\gamma G_i(y) = (0)$, for all x, y, m ∈ M and γ, β ∈ Γ. Thus we get $\sum_{i=1}^{n} D_i(m)\beta d_i(\alpha_i(x))\gamma G_i(y) = 0$, since M be semiprime then $d_n(\alpha_n(x))\gamma G_n(y) = 0$, for all x, y ∈ M and γ ∈ Γ …(1). Similar by hypothesis, we have $G_n(x)\gamma D_n(y) = 0$ and by same way above then $G_n(y)\gamma d_n(\alpha_n(x)) = 0$ …(2). Hence from (1) and (2) we get the required. The proof of (iii) is similar way in (ii). (iv) by (i) we have $\sum_{i=1}^{n} D_i(x\delta z)\gamma G_i(y\beta w) = 0$, for all x, y, m, z, w ∈ M and γ,δ,β∈Γ. Therefore we get $\sum_{i=1}^{n} D_i(x)\delta\alpha_i(d_i(z))\gamma G_i(y)\beta\alpha_i(g_i(w)) = 0$.

Since M is semiprime, then $\sum_{i=1}^{n} d_i(\alpha_i(z))\gamma G_i(y)\beta g_i(\alpha_i(w)) = 0$ which shows the required.

While proof (v) and (vi) then by using (ii) and (iv) we have $\sum_{i=1}^{n} d_i(x)\delta z\beta G_i(y) = 0$, for all x, y, z ∈ M and δ, β ∈ Γ. Hence $0 = \sum_{i=1}^{n} G_i(d_i(x)\delta z\beta G_i(y))$; therefore $0 = \sum_{i=1}^{n} G_i(d_i(x))\delta g_i(\alpha_i(z))\beta G_i(g_i(\alpha_i(\alpha_i(y))))$ (α_n is commu.) for all x, y, z ∈ M and δ, β

$\in \Gamma$. Replacing $g_i(\alpha_i(\alpha_i(y)))$ by $d_i(x)$ and M be semiprime then we get $G_n d_n = 0$. By similarly, since $\sum_{i=1}^{n} d_i(G_i(x)\delta z\beta d_i(y)) = 0$, $\sum_{i=1}^{n} g_i(D_i(x)\delta z\beta g_i(y)) = 0$ and $\sum_{i=1}^{n} D_i(g_i(x)\delta z\beta D_i(y)) = 0$ holds for all x, y, z $\in$ M and $\delta, \beta \in \Gamma$, we get $d_n G_n = 0$, $g_n D_n = 0$ and $D_n g_n = 0$ respectively. (vi) since $\sum_{i=1}^{n} G_i(D_i(x)\delta z\beta G_i(y)) = 0$ and $\sum_{i=1}^{n} D_i(G_i(x)\delta z\beta D_i(y)) = 0$ then $G_n D_n = D_n G_n = 0$.

Theorem (3.5): Let $D_n = (D_i)_{i \in N}$ and $G_n = (G_i)_{i \in N}$ be an α-derivations and acts as a generalized higher homomorphism mappings of a Γ-ring M, where $\alpha = (\alpha_i)_{i \in N}$ be a family commuting of endomorphisms of M. Then for all x, y $\in$ M and $\gamma, \beta \in \Gamma$ the following conditions are equivalent:

(i) D_n and G_n are orthogonal.

(ii) (a) $D_n(x)\gamma G_n(y) + G_n(x)\gamma D_n(y) = (0)$

(b) $d_n(\alpha_n(x))\gamma G_n(y) + g_n(\alpha_n(x))\gamma D_n(y) = (0)$, where in particular α_n be commuting mappings of M.

(iii) $D_n(x)\gamma G_n(y) = d_n(\alpha_n(x))\gamma G_n(y) = 0$, where α_n be commuting mappings of M.

(iv) $D_n(x)\delta G_n(y) = 0$, for all x, y $\in$ M and $d_n G_n = 0 = d_n g_n$.

Proof : **(i) $\Rightarrow$ (ii), (iii), (iv)** are proved by lemma (3.4).

Now **(ii) $\Rightarrow$ (i)** by (a) Replacing x by $m\beta x$ we get

$$\sum_{i=1}^{n} D_i(m)\beta\alpha_i(d_i(x))\gamma G_i(y) + G_i(m)\beta\alpha_i(g_i(x))\gamma D_i(y) = 0.$$

By lemma (3.3) $\sum_{i=1}^{n} D_i(m)\gamma G_i(y) = 0 = \sum_{i=1}^{n} G_i(y)\gamma D_i(m)$, for all x, y $\in$ M and $\gamma \in \Gamma$ that mean shows (i). Also **(iii) $\rightarrow$ (i)** Since $D_n(x)\gamma G_n(y) = 0$, replacing x by $\alpha\beta z$ we get $\sum_{i=1}^{n} D_i(x)\beta\alpha_i(d_i(z))\gamma G_i(y) = 0$, for all x, y, z $\in$ M and $\gamma, \beta \in \Gamma$. By lemma (3.3) then D_n, G_n are orthogonal. While **(iv) $\rightarrow$ (i)** since $d_n G_n = 0$, therefore $0 = \sum_{i=1}^{n} d_i G_i(x\beta y) = \sum_{i=1}^{n} G_i(d_i(x))\beta d_i(g_i(\alpha_i(\alpha_i(y))))$, replacing x by $x\delta z$ we get $\sum_{i=1}^{n} G_i(d_i(x))\delta g_i(d_i(\alpha_i(\alpha_i(z))))\beta d_i(g_i(\alpha_i(\alpha_i(y)))) = 0$ so by lemma (3.3) we get $\sum_{i=1}^{n} G_i(d_i(x))\gamma d_i(g_i(\alpha_i(\alpha_i(y)))) = 0$, therefore $\sum_{i=1}^{n} d_i(g_i(\alpha_i(\alpha_i(y))))\gamma G_i(d_i(x)) = 0$, replacing $g_i(\alpha_i(\alpha_i(y)))$ by $\alpha(y)$ and $d_i(x)$ by y and by theorem (3.5)(iii) we get D_n, G_n are orthogonal.

Lemma (3.6): Let $D_n = (D_i)_{i \in N}$ and $G_n = (G_i)_{i \in N}$ be a generalized higher α-derivation on Γ-ring M. Then D_n and G_n are orthogonal if there exist ideals U and V of M such that **(a)** $U \cap V = 0$ and $U \oplus V$ is essential ideal of M. **(b)** $D_n(M)$, $d_n(M) \subset U$ and $G_n(M)$, $g_n(M) \subseteq V$.
(c) $D_n(V) = d_n(V) = (0)$ and $G_n(U) = g_n(U) = 0$.

Proof : Let U_0 be the ideal of M generated by $d_n(M) \cup D_n(M)$. Let $Ann(U_0) = V$ and $Ann(V)=U$. By lemma (3.4) then we see that $D_n(x)\gamma G_n(y) = G_n(x)\gamma D_n(y) = 0$ and by $d_n(\alpha_n(x))\gamma G_n(y) = g_n(\alpha_n(x))\gamma D_n(y) = 0$, for all x, y $\in$ M and $\gamma \in \Gamma$. And by (iv) of lemma (3.4) we get $\sum_{i=1}^{n} d_i(x)\gamma g_i(y) = \sum_{i=1}^{n} g_i(y)\gamma d_i(x) = 0$, for all x, y $\in$ M and $\gamma \in \Gamma$. Since $D_n(M)$, $d_n(M) \subset U_0$ we obtain $G_n(V)$, $g_n(M) \subset V$ by lemma (3.4) and $U_0 \subset U$ we have $D_n(V) = d_n(V) = 0$ and $G_n(U) = g_n(U) = 0$, since M is semiprime then $U \oplus V$ is an essential ideal of M.

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Received: August 15, 2013