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On Flushed Partitions

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Abstract

A partition of the natural number n is said to be *flushed* if the least part that occurs evenly often is even. Let $f(n)$ denote the number of flushed partitions of n. We present explicit and recursive formulas for $f(n)$, as well as a table of $f(n)$ where $0 \leq n \leq 30$. We also discuss self-conjugate flushed partitions.

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1. Introduction

A partition of the natural number n is said to be *flushed* if the least part that occurs evenly often is even. (Zero occurrences counts as an even number.) This definition is due to Sylvester [4] , who also gave a conjecture concerning flushed partitions that is stated in (d) below. The proof of a generalization of Sylvester's conjecture was obtained by G. Andrews [1] using q-series; more recently, a combinatorial proof was found by C. Liu [3].

Let $f(n)$ denote the number of flushed partitions of the natural number n. (We also define $f(0) = 0$.) In this note, we derive an explicit formula for $f(n)$ in terms of the partition function, as well as an Euler-type recurrence for $f(n)$. We also present a table for $f(n)$, where $0 \leq n \leq 30$. We briefly discuss self-conjugate flushed partitions. Finally, we present as asymptotic conjecture concerning $f(n)$.

2. Preliminaries

Definitions Let z be a complex variable such that $|z| < 1$.

Let $\omega(k) = \frac{k(3k-1)}{2}$ where $k \in Z$ (k^{th} pentagonal number)

Let $p(n)$ denote the unrestricted partition function.

Let $f(n)$ denote the number of flushed partitions of n.

Identities, etc.

$$(a) \qquad \prod_{n=1}^{\infty}(1-z^n)^{-1} = \sum_{n=0}^{\infty} p(n)z^n$$

$$(b) \qquad \prod_{n=1}^{\infty}(1-z^n) = \sum_{n=-\infty}^{\infty} (-1)^n z^{\omega(n)}$$

$$(c) \qquad \sum_{n=1}^{\infty} f(n)z^n = \prod_{n=1}^{\infty}(1-z^n)^{-1} \sum_{n=1}^{\infty} z^{\omega(n)}(1-z^n)$$

(d) For all natural numbers, n, the number of unflushed partitions with oddly many parts equals the number of unflushed partitions with evenly many parts.

(e) $p(n)$ changes parity infinitely often as n tends to infinity.

Remarks: Identities (a) and (b) are well-known formulas in the theory of partitions; (c) appears in [3] ; (d) is Sylvester's conjecture; (e) was proved by Kolberg [2].

3. The Main Results

We begin with a theorem that is an easy consequence of work by other researchers, but that appears not to have been stated previously. Theorem 1 expresses $f(n)$ in terms of $p(n)$.

Theorem 1 For all $n \geq 1$, we have

$$f(n) = \sum_{k \geq 1}(p(n-\omega(k)) - p(n-\omega(-k)) \quad .$$

Proof: Identity (c) implies

$$\sum_{n=1}^{\infty} f(n)z^n = \prod_{n=1}^{\infty}(1-z^n)^{-1}\sum_{n=1}^{\infty}(z^{\omega(n} - z^{\omega(-n)})$$

The conclusion now follows from (a) , matching coefficients of like powers of z. ■

The next theorem is a recurrence concerning $f(n)$.

Theorem 2 For all $n \geq 1$, we have

$$\sum_{k=-\infty}^{\infty} (-1)^k f(n-\omega(k)) = \begin{cases} \pm 1 & \text{if} \quad n = \omega(\pm r) \\ 0 & \text{otherwise} \end{cases}$$

Proof: Identity (c) implies

$$\prod_{n=1}^{\infty}(1-z^n)\sum_{n=1}^{\infty} f(n)z^n = \sum_{n=-\infty}^{\infty} z^{\omega(n)} \ .$$

The conclusion now follows from (b), matching coefficients of like powers of z. ■

The table below lists $f(n)$ for $1 \leq n \leq 30$.

n	1	2	3	4	5	6	7	8	9	10
$f(n)$	1	0	1	1	3	3	5	6	10	12
n	11	12	13	14	15	16	17	18	19	20
$f(n)$	18	23	33	41	56	71	95	119	156	195
n	21	22	23	24	25	26	27	28	29	30
$f(n)$	252	314	399	495	624	768	958	1176	1455	1776

If we compare the values of $f(n)$ with corresponding values of $p(n)$, we obtain the following conjecture:

Conjecture As n tends to infinity, the ratio $f(n)/p(n)$ tends toward a constant whose approximate value is .316.

The next theorem concerns the parity of $f(n)$.

Theorem 3 $f(n)$ changes parity infinitely often as n tends to infinity.

Proof: Identity (d) implies the total number of unflushed partitions of n is even, hence $f(n) \equiv p(n)$ for all $n \geq 1$. The conclusion now follows from (e). ■

Finally, we present a theorem concerning flushed partitions that are self-conjugate.

Theorem 4 If $n \geq 21$, then there is at least one flushed, self-conjugate partition of n.

Proof: We will consider the congruence class of $n \pmod 4$.

If $n \equiv 0 \pmod 4$, let $n = 4j$, where $j \geq 6$. Then a flushed, self-conjugate partition of n is:

$$n = (2j-6) + 5^2 + 4 + 3 + 1^{2j-11} \quad .$$

If $n \equiv 1 \pmod 4$, let $n = 4j+1$, where $j \geq 4$. Then a flushed, self-conjugate partition of n is:

$$n = (2j-3) + 4^2 + 3 + 1^{2j-7} \quad .$$

If $n \equiv 2 \pmod 4$, let $n = 4j+2$, where $j \geq 3$. Then a flushed, self-conjugate partition of n is:

$$n = (2j) + 3 + 2 + 1^{2j-3} \quad .$$

If $n \equiv 3 \pmod 4$, let $n = 4j+3$, where $j \geq 1$. Then a flushed, self-conjugate partition of n is:

$$n = (2j+2) + 1^{2j+1} \quad . \quad ■$$

Remarks: There are no flushed self-conjugate partitions for $n = 2, 3, 4, 5, 8, 9, 10, 12, 13, 16, 19, 20$.

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