International Mathematical Forum, Vol. 8, 2013, no. 29, 1445 - 1454
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/imf.2013.37136

On Nearly-Kaehlerian Weyl Spaces

Fatma Özdemir and Mustafa Deniz Türkoğlu

Istanbul Technical University Faculty of Science and Letters
Department of Mathematics 34469 Maslak-Istanbul, Turkey
fozdemir@itu.edu.tr and mdturkoglu@hotmail.com

Abstract

In this work, we consider nearly-Kaehlerian Weyl spaces and show that a nearly-Kaehlerian Weyl space is a Kaehlerian if the almost complex structure is integrable. Also, we give a condition so that an almost semi-Kaehlerian structure to be Kaehlerian.

Mathematics Subject Classification: 53A30, 55A40

Keywords: Prolonged covariant derivative, nearly-Kaehlerian Weyl space, almost semi Kaehlerian-Weyl space

1. Introduction

Let W_n be an n-dimensional space with a conformal metric tensor g. If the torison-free connection D on W_n satisfies the compatibility condition

$$D_k g_{ij} = 2\, w_k\, g_{ij} \tag{1.1}$$

then, W_n is called a Weyl space and denoted by $W_n(g, w)$. Here, w is a 1-form, called complementary co-vector field.
In literature, it is shown that for the renormalization transformation of the metric tensor g

$$\widetilde{g}_{ij} = \lambda^2\, g_{ij} \tag{1.2}$$

the complementary vector field w is transformed into by the rule

$$\widetilde{w}_k = w_k + \partial_k \ln \lambda \tag{1.3}$$

where λ is a scalar function defined on W_n [3], [5], and [6].

A quantity A defined on $W_n(g, w)$ is called a satellite of g of weight $\{p\}$, if it admits a transformation of the form

$$\widetilde{A} = \lambda^p A \tag{1.4}$$

under the renormalization (1.2).
We note that the weight of the metric tensor g_{ij} is $\{2\}$ by the transformation (1.2).
The prolonged (extended) covariant derivative of the satellite A of weight $\{p\}$ is defined by [4], and [8]

$$\dot{\nabla}_k A = D_k A - p\, w_k\, A. \tag{1.5}$$

From (1.1), (1.2) and (1.5), and taking prolonged covariant derivative of the metric tensor g_{ij}, we have

$$\dot{\nabla}_k\, g_{ij} = 0. \tag{1.6}$$

Using the Ricci identity for a covariant vector field v_k , we write

$$(\dot{\nabla}_i \dot{\nabla}_j - \dot{\nabla}_j \dot{\nabla}_i)\, v_k = -W_{kji}{}^{t}\, v_t \tag{1.7}$$

then the curvature tensor of W_n is obtained as

$$W_{kji}{}^{l} = \partial_k \Gamma_{ji}^{\ l} - \partial_j \Gamma_{ki}^{\ l} - \Gamma_{ki}^{\ t} \Gamma_{jt}^{\ l} + \Gamma_{ji}^{\ t} \Gamma_{kt}^{\ l}. \tag{1.8}$$

Here, $\Gamma_{ji}^{\ l}$ are the coefficients of the Weyl connection

$$\Gamma_{ji}^{\ l} = \left\{ \begin{matrix} l \\ ji \end{matrix} \right\} - (w_j \delta_i^l + w_i \delta_j^l - g_{ij} w^l) \tag{1.9}$$

which is symmetric with respect to lower indices. In (1.9), $\left\{ \begin{smallmatrix} l \\ ji \end{smallmatrix} \right\}$ are the coefficients of the metric connection ∇.
The covariant curvature tensor, the Ricci curvature tensor and the scalar curvature of Weyl space are defined by respectively,

$$W_{kji}{}^{h} = W_{kjil}\, g^{lh}, \tag{1.10}$$

$$W_{kjil} = W_{kji}{}^{m}\, g_{ml}, \tag{1.11}$$

$$W_{ji} = g^{kl}\, W_{kjil} = W_{kji}{}^{k}, \tag{1.12}$$

$$W = g^{ji}\, W_{ji}. \tag{1.13}$$

It is also observed that the covariant curvature tensor W_{ijkl} of W_n satisfies the following relations [1]

$$1)\qquad W_{ijkl}+W_{jikl}=0, \tag{1.14}$$

$$2)\qquad W_{ijkl}+W_{ijlk}=2g_{kl}(w_{i,j}-w_{j,i}), \tag{1.15}$$

$$3)\qquad W_{lijk}+W_{ijlk}+W_{jlik}=0. \tag{1.16}$$

Furthermore, it can be seen that the anti-symmetric part of the Ricci curvature tensor satisfies the relation [6]

$$W_{[ij]}=n\nabla_{[i}w_{j]}\,. \tag{1.17}$$

The Ricci curvature tensor is not necessarily symmetric on W_n since the Weyl connection is not metric.

We here quote some definitions of [2], and [7].
Let W_n be a Weyl space of dimension $n=2m(m\geq 1)$. A tensor $F_i^{\ j}$ of type (1,1) with weight $\{0\}$ is called an almost complex structure on W_n if the tensor $F_i^{\ j}$ satisfies the condition

$$F_i^{\ j}\,F_j^{\ k}=-\,\delta_i^k, \tag{1.18}$$

and W_n is called an almost complex Weyl space. If W_n admits an almost complex structure $F_i^{\ j}$ satisfying the condition

$$g_{ij}\,F_h^{\ i}\,F_k^{\ j}=g_{hk}, \tag{1.19}$$

then $F_i^{\ j}$ is called an almost Hermitian structure with a Hermitian metric g on W_n. It is said that an almost Hermitian structure $F_i^{\ j}$ is a Kaehlerian structure (respectively, space) if $F_i^{\ j}$ satisfies the condition

$$\dot{\nabla}_k\,F_i^{\ j}=0 \qquad \text{for all}\ \ i,j,k, \tag{1.20}$$

then W_n becomes a Kaehlerian space which we denote by KW_n .

An almost Hermitian structure $F_i^{\ j}$ is called an almost Kaehlerian structure on W_n
(respectively, almost Kaehlerian Weyl space) if the tensor $F_i^{\ j}$ satisfies

$$F_{hij}=\dot{\nabla}_hF_{ij}+\dot{\nabla}_iF_{jh}+\dot{\nabla}_jF_{hi}=0. \tag{1.21}$$

An almost Hermitian structure $F_i^{\ j}$ is called an almost semi-Kaehlerian structure on W_n
(respectively, almost semi-Kaehlerian Weyl space denoted by SKW_n) if the tensor $F_i^{\ j}$ satisfies

$$F_i\equiv\dot{\nabla}_jF_i^{\ j}=0. \tag{1.22}$$

The contravariant and covariant tensors F_{ij} and F^{ij} are of weight $\{2\}$ and $\{-2\}$, respectively, and defined by means of metric tensor as

$$F_{ij} = g_{jk}\, F_i^{\ k}, \tag{1.23}$$

$$F^{ij} = g^{ih}\, F_h^{\ j}. \tag{1.24}$$

So it can be seen easily that,

$$F_{ij} = \ -F_{ji}, \tag{1.25}$$

$$F^{ij} = \ -F^{ji}. \tag{1.26}$$

The Nijhenuis torsion tensor of the Kaehlerian structure F_i^h on W_n with the weight $\{0\}$ is defined by

$$N_{ij}^{\ \ k} = F_i^{\ h}(\dot{\nabla}_h F_j^{\ k} - \dot{\nabla}_j F_h^{\ k}) - F_j^{\ h}(\dot{\nabla}_h F_i^{\ k} - \dot{\nabla}_i F_h^{\ k}), \tag{1.27}$$

and it is said that an almost complex structure is integrable on W_n if it has no torsion [2].

If we take the complementary vector field $w_k = 0$ in the prolonged covariant derivative in (1.5) then the above definitions reduce to those of Riemannian spaces [4, 8].

Now, we give some definitions of nearly-Kaehlerian structures and almost L-structures on W_n and some theorems concerning these structures.

2. Almost L-structures and nearly-Kaehlerian structures on Weyl spaces

An almost Hermitian structure $F_i^{\ j}$ of weight $\{0\}$ with a Hermitian metric g_{ij} on W_n satisfying

$$G_{ij}^{\ \ k} \equiv \dot{\nabla}_i F_j^{\ k} + \dot{\nabla}_j F_i^{\ k} = 0, \tag{2.1}$$

is called a nearly-Kaehlerian structure or an almost Tachibana structure (respectively, nearly-Kaehlerian Weyl space denoted by NKW_n).
An almost Hermitian structure $F_i^{\ j}$ of weight $\{0\}$ on W_n satisfying

$$[\dot{\nabla}_j, \dot{\nabla}_k]\, F_i^{\ h} \equiv (\dot{\nabla}_j \dot{\nabla}_k - \dot{\nabla}_k \dot{\nabla}_j) F_i^{\ h} = 0, \tag{2.2}$$

is called an almost L-structure and a Weyl space admitting an almost L-structure is called almost L-Weyl space denoted by LW_n.

Since an almost Hermitian structure $F_i^{\ j}$ on KW_n satisfies $\dot{\nabla}_k F_i^{\ j} = 0,$ for all i, j, k, then the Kaehlerian structure (respectively, the space KW_n) is an almost L-structure (respectively, the space LW_n).

Theorem 2.1. A nearly-Kaehlerian structure (respectively, the space NKW_n) is an almost semi-Kaehlerian Weyl structure (respectively, the space NKW_n).

Proof. In (2.1), contracting with respect to i and k gives

$$\dot{\nabla}_i F_j^{\ i} = -\dot{\nabla}_j F_i^{\ i}. \tag{2.3}$$

On the other hand, from (1.21), we get $F_i^{\ m} = F_{ij}\, g^{jm}$ and contracting with respect to m and i, we obtain $F_i^{\ i} = F_{ij}\, g^{ij} = 0$. Hence, it follows that $\dot{\nabla}_i F_j^{\ i} = 0$, which shows that the structure $F_j^{\ i}$ is an almost semi-Kaehlerian Weyl structure.

Theorem 2.2. A nearly-Kaehlerian Weyl space is a Kaehlerian-Weyl space if the structure is integrable.

Proof. Using the Nijhenuis torsion tensor of the nearly-Kaehlerian structure $F_i^{\ h}$ we obtain on W_n [1]

$$N_{ij}^{\ \ k} = F_i^{\ h}(\dot{\nabla}_h F_j^{\ k} - \dot{\nabla}_j F_h^{\ k}) - F_j^{\ h}(\dot{\nabla}_h F_i^{\ k} - \dot{\nabla}_i F_h^{\ k}), \tag{2.4}$$

and also using the defining condition (2.1) we obtain

$$\dot{\nabla}_h\, F_j^{\ k} = -\dot{\nabla}_j\, F_h^{\ k} \tag{2.5}$$

and

$$\dot{\nabla}_h\, F_i^{\ k} = -\dot{\nabla}_i\, F_h^{\ k}. \tag{2.6}$$

Replacing (2.6) in (2.4) we obtain

$$N_{ij}^{\ \ k} = -2F_i^{\ h}\, \dot{\nabla}_j F_h^{\ k} + 2F_j^{\ h}\, \dot{\nabla}_i F_h^{\ k}. \tag{2.7}$$

Since

$$F_j^{\ h}\, F_h^{\ k} = -\delta_j^k, \tag{2.8}$$

taking the prolonged covariant derivative of (2.8) gives

$$(\dot{\nabla}_i F_h^{\ k})\, F_j^{\ h} = -(\dot{\nabla}_i F_j^{\ h})\, F_h^{\ k}. \tag{2.9}$$

Similarly, taking the prolonged covariant derivative of

$$F_i^{\ h}\, F_h^{\ k} = -\, \delta_i^k, \tag{2.10}$$

yields

$$(\dot{\nabla}_j F_h{}^k)\, F_i{}^h = -(\dot{\nabla}_j F_i{}^h)\, F_h{}^k \tag{2.11}$$

and by means of (2.1) we obtain

$$(\dot{\nabla}_j F_h^k)\, F_i^h = -(\dot{\nabla}_j F_i^h)\, F_h^k = (\dot{\nabla}_i F_j^h)\, F_h^k. \tag{2.12}$$

On the other hand, substituting (2.9) and (2.12) in (2.4) we conclude that

$$N_{ij}{}^k = -4(\dot{\nabla}_i F_j^h) F_h^k. \tag{2.13}$$

By assumption, since the nearly-Kaehlerian structure is integrable, the Nijhenuis torsion tensor $N_{ij}{}^k$ is zero. Hence, we obtain

$$(\dot{\nabla}_i\, F_j{}^h) F_h^k = 0. \tag{2.14}$$

If we multiply (2.14) by F_k^m, and use (1.18) we thus obtain

$$\dot{\nabla}_i\, F_j^m = 0 \qquad \text{for all} \qquad i, j, m, \tag{2.15}$$

which gives that the structure $F_i{}^i$ is Kaehlerian.

We can express the defining condition (2.2) of an almost L-structure on LW_n in terms of the Weyl and Ricci curvature tensors.

Theorem 2.3. An almost Hermitian structure $F_i{}^j$ is an almost L-structure on LW_n if and only if

$$F_i{}^a W_{kja}{}^h = F_a{}^h W_{kji}{}^a. \tag{2.16}$$

Proof. Since the almost Hermitian structure $F_i{}^j$ is a tensor of weight $\{0\}$, we get

$$\dot{\nabla}_k \dot{\nabla}_j F_i{}^h - \dot{\nabla}_j \dot{\nabla}_k F_i{}^h = \nabla_k \nabla_j F_i{}^h - \nabla_j \nabla_k F_i{}^h, \tag{2.17}$$

and by using the Ricci identity [3] we have

$$(\dot{\nabla}_k \dot{\nabla}_j - \dot{\nabla}_j \dot{\nabla}_k) F_i{}^h = F_i{}^a W_{kja}{}^h - F_a{}^h W_{kji}{}^a. \tag{2.18}$$

If an almost Hermitian structure $F_i{}^j$ is L-structure on LW_n, then making use of (2.2) we obtain

$$(\dot{\nabla}_k \dot{\nabla}_j - \dot{\nabla}_j \dot{\nabla}_k) F_i{}^h = F_i{}^a W_{kja}{}^h - F_a{}^h W_{kji}{}^a = 0, \tag{2.19}$$

from which it follows that

$$F_i{}^a W_{kja}{}^h = F_a{}^h W_{kji}{}^a. \tag{2.20}$$

Conversely, suppose (2.16) holds. Then we have

$$F_i{}^a W_{kja}{}^h - F_a{}^h W_{kji}{}^a = \dot{\nabla}_k \dot{\nabla}_j F_i{}^h - \dot{\nabla}_j \dot{\nabla}_k F_i{}^h = 0, \tag{2.21}$$

which shows that the structure F_i^j is an almost L-structure on LW_n.

Theorem 2.4. On an almost Hermitian Weyl space condition (2.16) is equivalent to

$$W_{kijh} = F_j{}^l F_h{}^t W_{kilt}. \tag{2.22}$$

Proof. Suppose (2.16) holds. Multiplying (2.16) by $F_m{}^i$ gives

$$W_{kjm}{}^h = -F_a{}^h F_m{}^i W_{kji}{}^a. \tag{2.23}$$

First, multiplying (2.23) by g_{ht} and then using (1.23) and (1.25), respectively, we obtain

$$W_{kjmt} = -F_{at} F_m{}^i W_{kji}{}^a \tag{2.24}$$

$$W_{kjmt} = F_{ta} F_m{}^i W_{kji}{}^a, \tag{2.25}$$

which becomes

$$W_{kijh} = F_j{}^l F_h{}^t W_{kilt}. \tag{2.26}$$

Conversely, suppose that (2.22) holds then, multiplying (2.22) by $F_u{}^m$, we conclude that

$$F_u{}^m W_{kjmt} = -F_t{}^p W_{kjup}. \tag{2.27}$$

Multiplying g^{tn} and using (1.11) we obtain

$$F_u{}^m W_{kjm}{}^n = g^{pm} F_m{}^n W_{kjup}, \tag{2.28}$$

and

$$F_u{}^m W_{kjm}{}^n = F_m{}^n W_{kju}{}^m, \tag{2.29}$$

which implies that (2.16) holds.

Remark. It can be shown that the covariant curvature tensor W_{kjim} of W_n satisfies the following relation :

$$\begin{aligned} W_{kjim} - W_{imkj} &= g_{mk}(w_{i,j} - w_{j,i}) + g_{mj}(w_{k,i} - w_{i,k}) + g_{ji}(w_{m,k} - w_{k,m}) \\ &\quad + g_{ik}(w_{j,m} - w_{m,j}) + g_{im}(w_{k,j} - w_{j,k}) + g_{kj}(w_{i,m} - w_{m,i}). \end{aligned} \tag{2.30}$$

where $w_{i,j} - w_{j,i}$ stands for $w_{i,j} - w_{j,i} = \dot{\nabla}_j w_i - \dot{\nabla}_i w_j = \nabla_j w_i - \nabla_i w_j = \partial_j w_i - \partial_i w_j$.

We quote the following theorem from [2].

Theorem. If an almost semi-Kaehlerian structure F_i^j satisfies

$$\dot{\nabla}_k F_{ij} \dot{\nabla}^i F^{jk} = a \dot{\nabla}_k F_{ij} \dot{\nabla}^k F^{ij}, \tag{2.31}$$

where a is a non-zero constant, then Q defined by

$$Q \equiv W + \frac{1}{2} F^{ij} F^{kl} W_{ijkl} + 2g^{kj}(\nabla_j w_k - \nabla_k w_j) \tag{2.32}$$

is non-negative (non-positive) according as a is positive (negative).
Further, the structure $F_i^{\ j}$ is Kaehlerian on W_n if and only if $Q = 0$.
In [2], it is shown that

$$Q = a(\dot{\nabla}_k F_{ij})(\dot{\nabla}^k F^{ij}), \tag{2.33}$$

where $\dot{\nabla}^k = g^{jk} \dot{\nabla}_j$.
In particular, $Q = 0$ if and only if the structure is Kaehlerian on W_n.

The following theorem gives a relation between a nearly-Kaehlerian structure on NKW_n and a Kaehlerian structure on KW_n.

Theorem 2.5. For a nearly-Kaehlerian structure $F_i^{\ j}$ on NKW_n,

$$Q \geq 0, \tag{2.34}$$

where Q is given by (2.32) and the equality holds if and only the structure $F_i^{\ j}$ is Kaehlerian.

Proof. Since a nearly-Kaehlerian structure $F_i^{\ j}$ is an almost semi-Kaehlerian structure, multiplying (2.1) with g^{jm} we obtain

$$\dot{\nabla}_i F^{mk} + \dot{\nabla}^m F_i^{\ k} = 0 \tag{2.35}$$

and also multiplying (2.35) with g^{in} gives

$$\dot{\nabla}^n F^{mk} = -\dot{\nabla}^m F^{nk}. \tag{2.36}$$

Making use of (2.2), (2.33) and (2.36) we obtain

$$\dot{\nabla}_k F^{ij} (\dot{\nabla}^i F^{jk}) = (\dot{\nabla}_k F_{ij}) (-\dot{\nabla}^i F^{kj}) = (\dot{\nabla}_k F_{ij}) (\dot{\nabla}^k F^{ij}), \tag{2.37}$$

from which it follows that $a = 1$.
Also, we observe that the equality in (2.34) holds if and only if the structure $F_i^{\ j}$ is Kaehlerian.

Theorem 2.6. If an almost L-structure $F_i^{\ j}$ on LW_n is almost semi-Kaehlerian and satisfies either

$$\dot{\nabla}^k G_{hi}^{\ \ j} = 0 \tag{2.38}$$

or

$$\dot{\nabla}^k F_{hij} = 0, \tag{2.39}$$

where $\dot{\nabla}^k = g^{jk}\dot{\nabla}_j$, F_{hij} and $G_{hi}^{\ \ j}$ are defined in (1.21), and (2.1), respectively, then the structure $F_i^{\ j}$ is Kaehlerian.

Proof. From the defining condition (2.2) of an almost L-structure it follows that

$$\dot{\nabla}_k \, \dot{\nabla}_h \, F_i^{\ j} = \dot{\nabla}_h \, \dot{\nabla}_k \, F_i^{\ j}. \tag{2.40}$$

Multiplying (2.40) by g^{ik} gives

$$\dot{\nabla}^i \, \dot{\nabla}_h \, F_i^{\ j} = \dot{\nabla}_h \, \dot{\nabla}_k \, F^{kj} = -\dot{\nabla}_h \, \dot{\nabla}_k \, F^{jk} = -\dot{\nabla}_h \, \dot{\nabla}_k \, g^{ij} F_i^{\ k} = -g^{ij} \, \dot{\nabla}_h \dot{\nabla}_k \, F_i^{\ k}. \tag{2.41}$$

Since $F_i^{\ j}$ is an almost semi-Kaehlerian structure, it satisfies $\dot{\nabla}_k F_i^{\ k} = 0$ then (2.41) reduces to

$$\dot{\nabla}^i \, \dot{\nabla}_h \, F_i^{\ j} = 0. \tag{2.42}$$

i) Suppose (2.38) holds. Multiplying (2.38) by δ_k^i we obtain

$$\delta_k^i \, \dot{\nabla}^k \, G_{hi}^{\ \ j} = \delta_k^i \, \dot{\nabla}^k \, (\dot{\nabla}_h \, F_i^{\ j} + \dot{\nabla}_i \, F_h^{\ j}) = 0 \tag{2.43}$$

and using (2.42), (2.43) we find

$$\dot{\nabla}^i \, \dot{\nabla}_i \, F_h^{\ j} = 0. \tag{2.44}$$

On the other hand, since an almost Hermitian structure $F_i^{\ j}$ satisfying

$$F^{ij} \, \dot{\nabla}^k \, \dot{\nabla}_k \, F_{ij} = 0 \tag{2.45}$$

is Kaehlerian [1], thus (2.44) shows that the structure $F_i^{\ j}$ is Kaehlerian.

ii) Suppose (2.39) holds. Taking the prolonged covariant derivative of (2.39) gives

$$\dot{\nabla}^k \, F_{hij} = \dot{\nabla}^k \, \dot{\nabla}_h \, F_{ij} + \dot{\nabla}^k \, \dot{\nabla}_i \, F_{jh} + \dot{\nabla}^k \, \dot{\nabla}_j \, F_{hi} = 0. \tag{2.46}$$

Multiplying (2.46) by δ_k^i and using (1.23), respectively, we obtain

$$\dot{\nabla}^i \, \dot{\nabla}_h \, F_{ij} + \dot{\nabla}^i \, \dot{\nabla}_i \, F_{jh} + \dot{\nabla}^i \, \dot{\nabla}_j \, F_{hi} = 0 \tag{2.47}$$

and then

$$g_{jm} \dot{\nabla}^i \, \dot{\nabla}_h \, F_i^{\ m} + \dot{\nabla}^i \, \dot{\nabla}_i \, F_{jh} - g_{hm} \, \dot{\nabla}^i \, \dot{\nabla}_j \, F_i^{\ m} = 0. \tag{2.48}$$

By means of (2.42), (2.48) yields

$$\dot{\nabla}^i \, \dot{\nabla}_i \, F_{jh} = 0. \tag{2.49}$$

Using (2.44), from (2.49) it follows that the structure is Kaehlerian.

References

[1] EÖ. Canfes, A. Özdeğer, Some applications of prolonged covariant differentiation in Weyl space. , Journal of Geometry, 60(1997), 7-16.

[2] H. Demirbüker, F. Özdemir, Almost Hermitian, Almost Kahlerian and Almost Semi-Kahlerian Structures in Weyl Spaces, Buletinnul Sthntific Universitath Politehnica Din Timisoara Matematica-Fizica, Tom 43(57), No.2(1998), 1-7.

[3] L.P. Eisenhart, Non-Riemannian Geometry, New York Published by the American Mathematical Society 501 West 116th Street, 1927.

[4] L. Friedland, C.C. Hsiung C.C., A certain Class of almost Hermitian manifolds, Tensor, N.S. 48(1989), 252-263.

[5] V. Hlavaty, Theorie d'immersion d'une W_m dans W_n , Ann. Soc. Polon.Math., 21(1949), 196-206.

[6] A. Norden, Affinely connected spaces, GRFML, Moscow, 1976 (in Russian).

[7] F. Özdemir, G.Ç. Yıldırım, On Conformally Recurrent Kahlerian-Weyl Spaces, Topology and Its Applications 153 (2005), 477-484 .

[8] K. Yano, Differential Geometry on complex and Almost complex spaces, Pergamon Press, (1965).

Received: July 14, 2013