International Mathematical Forum, Vol. 8, 2013, no. 32, 1591 - 1596
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/imf.2013.3359

Existence of Two Conjugate Classes of A_5 within S_6 by Use of Character Table of S_6

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Abstract

In this paper, the existence of two classes of alternating group A_5 as a subgroup of symmetric Group S_n, for n = 6 is proved by using character table of S_6.

Key words: Symmetric Group, Character Table and Triangular Group

1 Introduction

The symmetric group S_n is defined over the regular figure n-gon with order n!. For n = 6 it has eleven conjugacy classes corresponding to partition P(6) which has been discussed in [1]. The symmetric group S_6 is, however a non-simple group of order $2^4 .3^2. 5$ but it contains a simple derived subgroup S_6' known as A_6 of index 2 of the order $2^3 .3^2 . 5$. The construction of character table of symmetric group S_6 has been studied in [2]. By [5], A_5 is the smallest non-abelian simple group of order $2^2.3.5$ and the smallest non-solvable group containing four conjuagacy classes.

A group $\Delta(2,3,k)$ of the form $\Delta(2,3,k) = < x, y \ ; \ x^2 = y^3 = (xy)^k = 1 >$ where k is any positive integer is known as triangular group [6]. For k = 5, it is isomorphic to the alternating group A_5. It is known by [4] that every finite alternating and symmetric group except $A_6, A_7, A_8, S_5, S_6 \ and \ S_8$ is a factor group of $\Delta(2,3,k)$.

We focus, in this paper, on the proof of the following theorem;

Theorem Let G be a symmetric group of degree 6. Then there exist two non-conjugate classes of simple groups isomorphic to A_5 involved by classes C_α, C_β and C_γ such that $\alpha^2 = \beta^3 = 1 = (\alpha \beta) = \gamma^5$ in S_6.

Proof Since $|S_6| = 2^4.3^2.5$ is divisible by $|A_5| = 2^2.3.5$, therefore by Lagrange's Theorem A_5 may be a candidate to exist within S_6 as a subgroup. In order to search for the possibility of the existence of A_5 within S_6, we need to know necessary information about conjugacy classes and character table of S_6. It is a well-known fact that A_5 is the smallest non-abelian simple group and isomorphic to a $\Delta(2,3,5)$ which is generated by elements x of order 2, y of order 3 and $(xy)^5 = 1$.

$$\text{i.e. } \Delta < 2,3,5 > = \{< x, y >: x^2 = y^3 = 1 = (xy)^5\}.$$

From [3] if α and β are two class representatives of classes $C_\alpha \ and \ C_\beta$ of the orders 2 and 3 respectively and their product $\alpha\beta = \gamma$ is an element of the only class of order 5 in S_6, then

$$\# < \alpha.\beta = \gamma > = \frac{|S_6|}{|C_G(\alpha)||C_G(\beta)|} \sum_{i=1}^{11} \frac{\chi_i(\alpha)\chi_i(\beta)\overline{\chi_i(\gamma)}}{\chi_i(1)} \qquad (1)$$

where $\# < \alpha.\beta = \gamma >$ gives the number of solutions of equation $\alpha.\beta = \gamma$ (known as class constants), $\chi_i(1)$ = the degree of characters of G and $\chi_i(\alpha)$, $\chi_i(\beta)$ stand for character values of i^{th} character of corresponding conjugacy classes and $\overline{\chi_i(\gamma)}$ is conjugate of the character value of class representation of γ.

Character Table of S_6

Class Representatives	1	(12)	(123)	(1234)	(12345)	(123456)	(12)(34)	(12)(34)(56)	(12)(345)	(12)(3456)	(123)(456)
Class Size / *Characters*	1	15	40	90	144	120	45	15	120	90	40
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	-1	1	-1	-1	1	1
χ_3	5	3	2	1	0	-1	1	-1	0	-1	-1
χ_4	5	-3	2	-1	0	1	1	1	0	-1	-1
χ_5	9	3	0	-1	1	0	1	3	0	1	0
χ_6	9	-3	0	1	1	0	1	-3	0	1	0
χ_7	5	-1	-1	1	0	0	1	3	-1	-1	2
χ_8	5	1	-1	-1	0	0	1	-3	1	-1	2
χ_9	10	2	1	0	0	1	-2	-2	-1	0	1
χ_{10}	10	-2	1	0	0	-1	-2	2	1	0	1
χ_{11}	16	0	-2	0	1	0	0	0	0	0	-2

It is noted from the character table of S_6 that;

(i) There are three classes of elements of order 2 of the permutation types 2_α = (12), 2_β = (12)(34) *and* 2_γ = (12)(34)(56) respectively.

(ii) There are two classes of elements of order 3 of the permutation type 3_α = (123) and 3_β = (123)(456).

(iii) There is only one class of the elements of order 5 as 5_α of the type (12345).

By the use of character table of S_6 in (1) the following table of class coefficients reveals that only even permutations are involved in existence of $\Delta(2,3,5)$;

	$\beta = (123)$ $\lvert C_{S_6}(\beta)\rvert = 18$	$\beta = (123)(456)$ $\lvert C_{S_6}(\beta)\rvert = 18$	
$\alpha = (12)$ $\lvert C_{S_6}(\alpha)\rvert = 48$	0	0	$\gamma = (12345)$ $\lvert C_{S_6}(\gamma)\rvert = 5$
$\alpha = (12)(34)$ $\lvert C_{S_6}(\alpha)\rvert = 16$	5	5	$\gamma = (12345)$ $\lvert C_{S_6}(\gamma)\rvert = 5$
$\alpha = (12)(34)(56)$ $\lvert C_{S_6}(\alpha)\rvert = 48$	0	0	$\gamma = (12345)$ $\lvert C_{S_6}(\gamma)\rvert = 5$

Table (a)

Thus for each of relation # < 2.3 = 5 > = 5. These class coefficients ensure that there exist five triangular relations in S_6. These triangular relations form alternating group A_5 within S_6. Now the only question is left to be answered here that how many of such A_5 are conjugate? For this, we see that $\lvert C_{S_6}(\gamma)\rvert = 5$ where γ is a cyclic group of order 5. As α and β are two elements of orders 2 and 3 in S_6 within their specified conjugacy classes 2_α *and* 3_α respectively and $\gamma = \alpha\beta$ is of

order 5 then for any element c of order 5 in $C_{S_6}(\gamma)$. We have c (αβ) $c^{-1} = c_1\gamma\, c^{-1} = \gamma$, implies that c (α) c^{-1} $c\beta c^{-1} = \gamma$, thus $(c\alpha\, c^{-1})$
$(c\beta c^{-1}) = \gamma$. The same γ is produced by conjugating α and β with an element c of order 5 in $C_{S_6}(\gamma)$. So total number of pairs of α and β shall count to be equal in number to order of the centralizer of γ. Thus all the 5 relations become conjugate by conjugating α and β by the centralizer of γ. This concludes that all copies of A_5 stand conjugate to each other, which shall form a single conjugate class within S_6.

This concludes that there are just two non-conjugate classes of simple group $\Delta(2,3,5) \cong A_5$ within S_6.

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Received: March 11, 2013