International Mathematical Forum, Vol. 8, 2013, no. 30, 1487 - 1492
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/imf.2013.37141

On Multiplication Lattice Modules

C. S. Manjarekar

Department of Mathematics
Shivaji University
Kolhapur, India
csmanjrekar@yahoo.co.in

U. N. Kandale

Sharad Institute of Techology
Yadrav, Ichalakaranji, India
ujwalabiraje@gmail.com

Abstract

In this paper, we find some basic results of Multiplication Lattice Modules.

Keywords: Multiplicative lattice, lattice module, maximal element, prime element, primary element

1 Introduction

Multiplicative lattices are studied by R P Dilworth [3] and D D Anderson [1]. A multiplicative lattice L is a complete lattice provided with commutative, associative and join distributive multiplication in which the largest element I acts as a multiplicative identity. An element $a \in L$ is called proper if $a < 1$. A proper element p of L is said to be prime if $ab \leq p$ implies $a \leq p$ or $b \leq p$. An element $m < 1$ is called maximal if $m < x \leq 1$ implies $x = 1$. If $a \in L$,$b \in L$, $\alpha(a : b)$ is the join of all elements c in L such that $cb \leq a$. A proper element p of L is said to be primary if $ab \leq p$ implies $a \leq p$ or

$b^n \leq p$ for some positive integer n. If $a \in L$, the radical of a denoted by $\sqrt{a} = \vee\{x \in L \mid x^n \leqslant a, n \in Z_+\}$. An element $e \in L$ is called meet principal if $(a \wedge (b : e))e = ae \wedge b$, for all $a, b \in L$. An element $e \in L$ is called join principal if $(a \vee be) : e = (a : e) \vee b$ for all $a, b \in L$ and an element $e \in L$ is called principal if it is both meet and join principal. An element $a \in L$ is called compact if $a \leqslant \underset{\alpha}{\vee} b_\alpha$ implies $a \leqslant b_{\alpha 1} \vee b_{\alpha 2} \vee .... \vee b_{\alpha n}$ for some finite subset $\{\alpha_1, \alpha_2, ...\alpha_n\}$

If each element of L is the join of compact elements then L said to be compactly generated lattice (CG - Lattice) and if each element of L is the join of principal elements then L said to be principally generated lattice (PG - Lattice). A multiplicative lattice L called r-lattice if it is modular, principally generated, compactly generated and in which the largest element I is compact.

Let M be a complete lattice and L be a multiplicative lattice then M is called L-module or module over L if there is a multiplication between elements of L and M written as aB where $a \in L$ and $B \in M$ which satisfies the following properties,

1. $(ab)B = a(bB)$
2. $(\underset{\alpha}{\vee} a_\alpha)(\underset{\beta}{\vee} B_\beta) = \underset{\alpha,\beta}{\vee} a_\alpha B_\beta$
3. $IB = B$
4. $OB = O_M,\ for\ all\ a, a_\alpha\ , b \in L\ and\ B, B_\beta \in B$, where I is the supremum of L and O is the infimum of L.We denote by 0_M and I_M the least element and the greatest element of M.

Let M be a L-module. If $N \in M$ and $a \in L$ then $(N : a) = \vee\{X \in M \mid aX \leqslant N\}$. If $a, b \in L$, we write $(a : b) = \vee\{x \in L \mid bx \leqslant a\}$ If $A, B \in M$, then $(A : B) = \vee\{x \in L \mid xB \leqslant A\}$. An L-module M is called a multiplication L-module if for every element $N \in M$ there exists an element $a \in L$ such that $N = aI_M$. An element N of M is called meet principal if $(a \wedge (B : N))N = aN \wedge B$ for all $a \in L$ and for all $B \in M$. An element N of M is called join principal if $a \vee (B : N) = (aN \vee B) : N$ for all $a \in L$ and for all $B \in M$ and N is said to be principal if it is both meet principal and join principal.

A proper element N of M is said to be prime if $aX \leqslant N$ implies $X \leqslant N$ or $aI_M \leqslant N$ that is $a \leqslant (N : I_M)$ for every $a \in L$, $X \in M$. If N is prime element of M then $(N : I_M)$ is prime element of L [4](proposition (3.6)). An element $N < I_M$ in M is said to be primary if $aX \leqslant N$ implies $X \leqslant N$ or $a^n I_M \leqslant N$ that is $a^n \leqslant (N : I_M)$ for some integer n. If $(O_M : I_M) = O$ then M is called faithful L-Module. For $B \in M$, $\sqrt{B} = \vee\{a \in L \mid a^n I_M \leqslant B\}$ for some positive integer n.If each element of M is the join of principal (compact) elements of M then M is called principally generated (compactly generated) lattice. Our multiplicative lattice L will be an r-lattice and lattice module M will be a faithful multiplication PG-Lattice (Principally generated) L-Module.

2 Prime element, primary elements and radicals

Free multiplication modules are studied by Saeed Rajaee [6]. In his paper, he gave the relation between prime and primary ideals of a ring R and the corresponding prime and primary submodules in a module over R. The next theorem gives the relation between a prime element in a multiplicative lattice and a prime element in a lattice module. Similarly, we obtain the relation for primary element in L and primary element in M.

Theorem 2.1. *Let M be a lattice module and p be a prime element of L and q be a primary element of L then* $(I_M : p)$ *is a prime element of M and* $(I_M : q)$ *is a primary element of M. Morever for element a and b of L,* $aI_M \leqslant bI_M$ *implies* $a \leqslant b$ *if and only if* $(aI_M : I_M) = a$.

Proof. Let p be a prime element of L and $aX \leqslant (I_M : p) = Q$. Suppose, $X \nleqslant (I_M : p)$, let $Y \leqslant aI_M$. We show that, $Y \leqslant (I_M : p)$ or equivalently, $pY \leqslant I_M$ which is obvious. Hence, $aI_M \leqslant (I_M : p)$ and hence, $(I_M : p)$ is prime element of M. Next, let q be a primary element of L and $aX \leqslant (I_M : q)$. Suppose, $X \nleqslant (I_M : q)$ and $Y \leqslant a^n I_M$. We show that, $Y \leqslant (I_M : q)$ or equivalently, $qY \leqslant I_M$ which is obvious. Hence, $a^n I_M \leqslant (I_M : q)$ and $(I_M : q)$ is primary element of M. Suppose, $aI_M \leqslant bI_M$ implies $a \leqslant b$, we have $a \leqslant (aI_M : I_M)$. Let $Y \in S = \{x \in L \mid xI_M \leqslant aI_M\}$. Therefore, $YI_M \leqslant aI_M$ and hence, $Y \leqslant a$ [2]. This shows that, $(aI_M : I_M) = a$. Conversely, let $(aI_M : I_M) = a$. If $aI_M \leqslant bI_M$ then $(aI_M : I_M) \leqslant (bI_M : I_M)$ and hence, $a \leqslant b$. □

In the next result, we obtain another presentation of radical of an element A of M.

Theorem 2.2. *Let M be a multiplication L-Module and element* $A \in M$. *Then,* $\sqrt{A} = \sqrt{q}$ *where* $A = qI_M$.

Proof. We have, $\sqrt{A} = \vee\{a \in L \mid a^n I_M \leqslant A, \; n \in z_+\}$ [1]. Let $A = qI_M$ for some $q \in L$. Then, $\sqrt{A} = \vee\{a \in L \mid a^n I_M \leqslant qI_M, \; n \in z_+\} = \vee\{a \in L \mid a^n \leqslant q \;, \; n \in z_+\}$. So, $\sqrt{A} = \sqrt{q} \;, \; q \in L$. □

We prove some elementary properties of radicals. For properties of radicals in a multiplicative lattice one can refer N K Thakare and C S Manjarekar [7].

Theorem 2.3. *Let M be a multiplicative L-Module where for any element A and B of M,*

1. $\sqrt{\sqrt{A}} = \sqrt{A}$
2. $\sqrt{A} \vee \sqrt{B} \leqslant \sqrt{A \vee B}$
3. $If \quad A \vee B = I_M \quad then \quad \sqrt{A \vee B} = I_L$
4. $\sqrt{A \vee B} = \sqrt{\sqrt{A} \vee \sqrt{B}}$
5. $\sqrt{A \wedge B} = \sqrt{A} \wedge \sqrt{B}$ *,if multiplication distributes over meet.*

Proof. Let $A = aI_M \quad and \quad B = bI_M$ for some elements a and b of L, then by theorem (2.2) $\sqrt{A} = \sqrt{a}$, $\sqrt{B} = \sqrt{b}$. Now, we have,

1. $\sqrt{\sqrt{A}} = \sqrt{\sqrt{a}} = \sqrt{a} = \sqrt{A}$.
2. We have, $\sqrt{A} \vee \sqrt{B} = \sqrt{a} \vee \sqrt{b}$; $A \vee B = (a \vee b)I_M, \sqrt{A \vee B} = \sqrt{a \vee b}$, $\sqrt{a} \vee \sqrt{b} = [\vee\{x \in L \mid x^n \leqslant a\}] \vee [\vee\{x \in L \mid x^n \leqslant b\}]$. Let $x \in S \cup S^{'}$ *where* $S = \{x \in L \mid x^n \leqslant a\}$ *and* $S^{'} = \{x \in L \mid x^n \leqslant b\}$.*So* $x \in S$ *or* $x \in S^{'}$. *Then,* $x^n \leqslant a$ *or* $x^m \leqslant b$, $n, m \in Z_+, n > m$. *Hence,* $x^n \leqslant a \vee b$ *and* $x \in \{y \mid y^n \leqslant a \vee b\} = S_1$. Therefore, $\sqrt{a} \vee \sqrt{b} \leqslant \sqrt{a \vee b}$. *This shows that,* $\sqrt{A} \vee \sqrt{B} = \sqrt{a} \vee \sqrt{b} \leqslant \sqrt{a \vee b} = \sqrt{A \vee B}$.
3. Let, $A \vee B = I_M$. Then, $aI_M \vee bI_M = (a \vee b)I_M = I_M$. So, $a \vee b = I_L$. Hence, $\sqrt{a \vee b} = \sqrt{I_L}$ that is $\sqrt{A \vee B} = I_L$
4. We have, $\sqrt{A \vee B} = \sqrt{aI_M \vee bI_M} = \sqrt{(a \vee b)I_M} = \sqrt{a \vee b}$. We know that, $\sqrt{a \vee b} = \sqrt{\sqrt{a} \vee \sqrt{b}}$. Therefore, $\sqrt{A \vee B} = \sqrt{\sqrt{A} \vee \sqrt{B}}$.
5. We have, $\sqrt{A} \wedge \sqrt{B} = \sqrt{a} \wedge \sqrt{b}$, $A \wedge B = aI_M \wedge bI_M = (a \wedge b)I_M$ (Since, multiplication distributes over meet). But, $\sqrt{A \wedge B} = \sqrt{a \wedge b} = \sqrt{a} \wedge \sqrt{b}$. Therefore, $\sqrt{A \wedge B} = \sqrt{A} \wedge \sqrt{B}$.

The next theorem gives the relation between primary element of lattice module M and prime element of multiplicative lattice L. □

Theorem 2.4. *Let M be a multiplication L-module, where for any element A of M, if Q is primary element of M then* $\sqrt{Q}$ *is prime element of L.*

Proof. Let $Q = qI_M$, for some element q of L. Now, $(Q : I_M) = (qI_M : I_M) = q$ is a primary element of L. Hence, $\sqrt{q}$ is a prime element of L. Therefore, by theorem (2.2), $\sqrt{Q} = \sqrt{q}$ is prime element of L. □

The next result gives the characterization for elements to be equal.

Theorem 2.5. *Let N_1 and N_2 be two elements of multiplication lattice module M. Then $(N_1 : I_M) = (N_2 : I_M)$ if and only if $N_1 = N_2$.*

Proof. Let,$N_1 = aI_M$ *and* $N_2 = bI_M$ for some element a, b *of* L.Let,$(N_1 : I_M) = (N_2 : I_M)$.Then $(aI_M : I_M) = (bI_M : I_M)$.Hence, a = b and $aI_M = bI_M$.Conversely, let $N_1 = N_2$ .So,$aI_M = bI_M$ gives $(aI_M : I_M) = (bI_M : I_M)$ and therefore, $(N_1 : I_M) = (N_2 : I_M)$. □

One may ask the question whether the prime element is minimal or not.The answer is given in the next theorem.

Theorem 2.6. *Let N be a prime element of multiplication L-Module M and p is the prime element of L and $(N : I_M) > p$, then N is not a minimal prime element of M.*

Proof. Since, M is a multiplication L module, $N = qI_M$ for some element q of L. We have, $(N : I_M) = (qI_M : I_M) = q$ is the prime element of L. Let p be a prime element in L. $p = (pI_M : I_M) < (N : I_M)$. Therefore, pI_M is a prime element of M contained in N [2] and hence, N is not a minimal prime element of M. □

The next result follows immediately.

Theorem 2.7. *Let M be a multiplication L-module and N be a minimal prime element of M then there is no prime element p of L where $(N : I_M) \geqslant p$.*

The next theorem gives a property of local multiplicative lattice.

Theorem 2.8. *Let (L, m) be a local multiplicative lattice and M be a multiplication L-Module such that $I_M \neq mI_M$ then mI_M is a maximal prime element of M.*

Proof. Since, m is maximal element of L then mI_M is maximal element of M and hence, it is a prime element [4]. Let N be prime element such that $N \geqslant mI_M$. Then, $(N : I_M) \geqslant (mI_M : I_M) = m$. This implies $N = mI_M$ Thus, mI_M is a maximal prime element of M. □

References

[1] Anderson D D,PhD Thesis.

[2] Callialp F and Tekir U, Multiplication lattice modules, Iranian Journal of Science and Technology, IJST (2011), A4, 309 - 313.

[3] Dilworth R P, Abstract commutative ideal theory, Pacific J Math, 12(1962), 481-498.

[4] Johnson J A 1970, a - adic completion of noetherian lattice module, Fund Math, 66, 341-371.

[5] Khouja-AI Eaman A, Maximal element and prime element in lattice modules, Damascus University Journal for Basic Sciences, Vol. 19, No 2, 2003.

[6] Saeed Rajee, Some remarks on free multiplication modules, International Journal of Algebra, Vol. 5, 2011 , no. 14, 655-659.

[7] Thakare N K, Manjarekar C S, Radicals and uniqueness theorems in multiplicative lattices with chain conditions, Studia Scientiarum Mathematicarum Hungarica, 18 (1983), 13-19.

Received: July 11, 2013