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Devaney's Chaos of One Parameter Family of Semi-triangular Maps

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Abstract

In [1] , it is proved that the family of semi-triangular maps

$$F = \{f_\alpha(x) = \alpha\ x\ \sin x : \alpha > 0\ ,\ \ x \epsilon R\}$$

has sensitive dependence on initial conditions (SDIC) for all $\alpha > 2$. In this work, we complete the two conditions: topologically transitive and density of periodic points in order to prove that F has a strong chaos (Devaney's chaos) .

Keywords: Devaney's chaos, strong chaos

1. Introduction

There are many definitions of chaos ranging from measure theoretic notions of randomness in ergodic theory to the topological approach. In fact the term"chaos" was first used by Li and York in a connection with a map without giving any formal definition [6]. Today, there are various definitions of a chaotic map like Shaihai definition [7] , Li-york [5] $\mathcal{W}$-chaos [4] , [7] and others .

In the topological dynamics, chaos is widely studied and the most used definition of chaos due to Devaney [3]. Devaney's chaos is also called strong chaos. A map f on a metric space X is chaotic, in the sense of Devaney, if it satisfies three conditions : SDIC , Topologically transitive and the set of periodic points of the map is dense in X . Recall that a map f:X $\to$ X on a metric space X is called topologically transitive if for any two open sets U , $V \subset$ X , there exists $k > 0$ Such that $f^{(k)}(U) \cap V \neq \emptyset$ $(f^{(k)} = fofo \ldots of \;\; \mathrm{k-times})$. And a map f is SDIC if there exists $\delta > 0$ such that for any $x \epsilon X$ and any neighborhood N of x there exists $y \in X$ and a natural number n such that $|f^{(n)}(x) - f^{(n)}(y)| > \delta$. In fact , Devaney's definition of chaos applies to a large number of important examples and in many cases it is easy to verify . Moreover , J.Banks in [2] , was proved that if a map f is transitive and has dense set of periodic points then f is SDIC .

In this work we prove that the functions of the family $F = \{f_\propto(x) = \propto x \sin x : \propto > 0 , x \in R\}$ are topologically transitive and have dense sets of periodic points in R .
We will need the following results which we proved in [1].

Theorem 1.1: Let $F = \{f_\alpha(x) = \alpha x \sin x \; ; x \in R , \alpha > 0\}$ then :
1. For $\alpha = 1$,the function $f_\alpha \in F$ has infinite number of fixed points [one fixed point in $(2n\pi, 2(n+1)\pi) \; \forall n = 0,1,2, \ldots$].
2. For $\alpha > 1$ the number in (1) is doubled .

Theorem 1.2: For $\alpha > 2$, the functions $f_\alpha \in R$ are sensitive dependence on initial conditions .

2. Devaney's chaos of the family F .

Let $F = \{f_\alpha(x) = \alpha x \sin x \; ; x \in R , \alpha > 0\}$ .In this section we prove that $f_\alpha \in F$ is topologically transitive and the set of periodic points is dense in R . We start with the following proposition :

Proposition 2.1: Let $f_\alpha \in F$. then :
1. The function $f_\alpha^{(n)}$ is continuous on R $\forall n = 1,2, \ldots$
2. For $m > n$, the number of periodic points of f of period m is greater than the number of periodic points of period n .

3. For any two intervals $I, J = (v_1, v_2)$ $in\ R$ $\exists m \in N$ such that $\max_{x \in I} f^{(m)}(x) > v_2$ (where $\max_{x \in I} f^{(m)}(x)$ is the maximum value of $f(x)$ on I).

Proof :

1. Clear .

2. Let I be an interval in R . If we derivative the function $f^{(n)}(x)$ and solve the equation $\frac{d f^{(n)}(x)}{dx} = 0$, then we obtain a number of local maximum and local minimum values of $f^{(n)}$. Moreover , these numbers are decreasing as n decreases, and for any open interval $I \subset R$, $\max_{x \in I} f^{(n+1)}(x) > \max_{x \in I} f^{(n)}(x)$. See figures (1,2,3).

But for all $n \epsilon N$, $f^{(n)}$ is continuous . Hence the graph of $f^{(n)}$ intersects the line $y = x$ at number of points . Moreover the graph of $f^{(m)}$ intersects the line $y = x$ at a number of points greater than the graph of $f^{(n)}$ does , for $m > n$. Therefore the number of periodic points of period m is greater than the number of periodic points of period n .

3. Let $I, J = (v_1, v_2)$ be two intervals in R . If $\max_{x \in I} f(x) > v_2$ then the result is true .

If $\max_{x \in I} f(x) \leq v_2$, then we test $\max_{x \in I} f^{(2)}(x)$ (note that , $\max_{x \in I} f^{(2)}(x) > \max_{x \in I} f(x)$ -See the proof of 2-) . If $f^{(2)}(x) \leq v_2$ then we test $f^{(3)}(x)$ and so on , until we find $m \in N$ such that $\max_{x \in I} f^{(m)}(x) > v_2$.

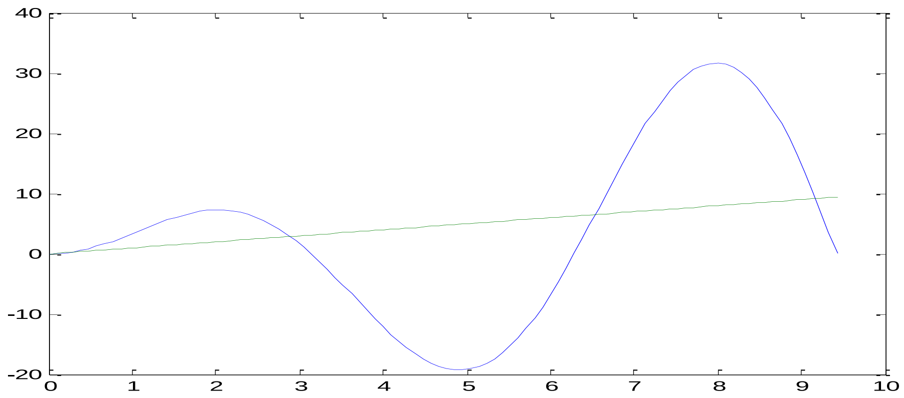

Figure -1-Graph of f_α^n, n=1,α=4

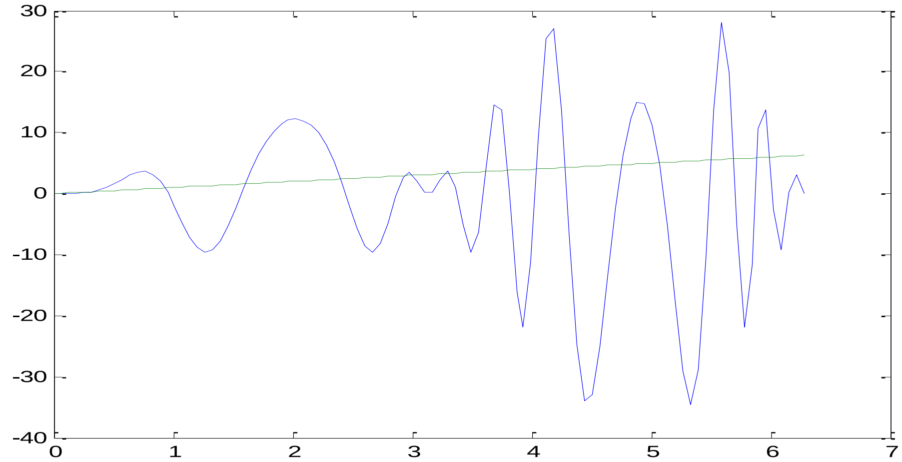

Figure-2-Graph of f_{α}^{n},n=2,α=4

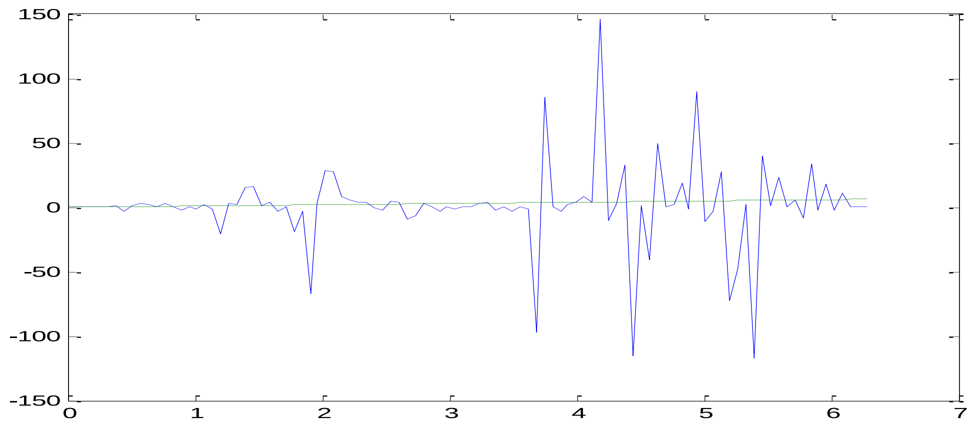

Figure-3-Graph of f_{α}^{n},n=5,α=4

Theorem 2.2 : Let $F = \{f_\alpha(x) = \alpha x \sin x \; ; x \in R \,, \alpha > 0\}$. Then we have :
1. Every $f_\alpha \in F$ is topologically transitive .
2. The set of periodic points of $f_\alpha \in F$ is dense in R.

Proof :

1. Let I and $J=(v_1, v_2)$ be two open intervals in R . To prove that $f_\alpha \in F$ is topologically transitive , we have to show that $f^{(m)}(I) \cap J \neq \emptyset$ for some $m \in N$.

a. If $f(I) \cap J \neq \emptyset$ then we done .

b. If $f(I) \cap J \neq \emptyset$, then by proposition 2.1 there exists $m \in N$ such that $\max_{x\in I} f^{(m)}(x) > v_2$.

But $f^{(m)}$ is continuous . Thus $f^{(m)}(I) \cap J \neq \emptyset$ and the prove is complete .

2.Let D be the set of periodic points of f_α . We have to show that D is dense in R , i.e. $\forall$ open interval $I \subseteq R$,$\exists\, p \in D$ s.t. $p \in I$.

Let $J_k = (k\pi,(k+2)\pi)$ $k = 0,1,2,\dots$. Without lose of generality we can assume $I \subset J_0 = (0,2\pi)$

Let $m \in N$ be a large number . Consider $f^{(n)}(x)$. By theorem (1.1) and proposition (2.1) f has a number periodic point of period n say $p_1, p_2, \dots, p_l$. If $I \cap D = \emptyset$, i.e. $I \subset (p_i, p_{i+1})$, then chose $m = n+1$.

Now , $\forall i$,$\exists\, p_{im} \in (p_i, p_{i+1})$ (see proposition (2.1)) , $p_{im} \in D$.

If $I \cap [(p_i, p_{im}) \cup (p_{im}, p_{i+1})] = \emptyset$,then chose $m = n+2$, and so on , until find p_i^* a periodic point and $p_i^* \in (p_{im}, p_{im+1}) \cap I$, that is p_i^* is a periodic point lies in I . Thus $I \cap D \neq \emptyset$ and D is dense in R , and the theorem is complete

Finally , if we companied the results in theorem (2.2) with theorem (1.2) , then we proved the following theorem which is the main result in our work :

Theorem (2.3) : Let $F = \{f_\alpha(x) = \alpha x \sin x \text{ }, x \in R, \alpha > 0\}$ .Then $f_\alpha \in F$ is Devany's (strong) chaotic .

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