International Mathematical Forum, Vol. 8, 2013, no. 31, 1531 - 1537
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/imf.2013.37148

Presentation of Semidirect Product of Monogenic Semigroups

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Abstract

By studying the semigroup presented by

$$\pi = \langle A, B \mid A^{n+1} = A, B^{m+1} = B, BA = A^{n-1}B, B^m = A^n \rangle,$$

for every positive integers $m, n \geq 3$ we show that, for even values of m this is an appropriate presentation for the semidirect product of monogenic semigroup $S = \langle a \mid a^{n+1} = a \rangle$ by the monogenic semigroup $T = \langle b \mid b^{m+1} = b \rangle$. Moreover, for odd values of m it is a commutative semigroup of order $\frac{(3+(-1)^n)m}{2}$. The interests of presentability of products of semigroups are quite useful in the study of more effective and intrinsic properties of the products (like as the computation of their characters and studying their Green J-classes). Our method of proof based on using the theoretic definitions and hereby we will deduced the presentation of the direct product of monogenic semigroups for every integers $n, m \geq 3$.

Mathematics Subject Classification: 20M30, 20E45

Keywords: monogenic semigroups, semidirect product of semigroups

1 Introduction

We may construct new semigroups of the existing semigroups by different definitions of products. The direct product of semigroups studied by certain authors, for instance, Tamura [11] gives examples of such structures and Robertson [10] studies the existence of direct product of semigroups by concerning the finite presentation in terms of generators and relators, where a presentation $\langle X \mid R\rangle$ for a semigroup consists of formal generators X and relators R, (for a detailed information on the semigroup presentation one may see [1,9], for examples.) However, the study of semidirect product of semigroups during the years 1961 by 2003, focuses on the general behaviors of special types of semigroups. All of these attempts consider the definition of the semidirect product of Clifford [3] and we would like to give here a short history of these nice attempts. Cartino [2] studies a property of semidirect products that involves the idempotent elements of semigroups. Pin [7] studies the semidirect product of ordered semigroups. Preston [8] characterizes the semidirect products that may be reduced to one of the three algebraic structures: group, regular semigroup or inverse semigroup. Among all of these researches which could be quite useful for applications of semigroup, one may consult MacAlister [6], and also, since the presentability of semigroups facilitates the applications of semigroups the articles Hosseinzadeh [5] and Garibkhajeh [4] focus on calculating the characters and Green graphs of finite semigroups.

In this paper which is the first attempt to give a finite presentation for semidirect products of semigroups, we find a finite presentation for the semidirect product of monogenic semigroup $S = \langle a \mid a^{n+1} = a\rangle$ by the monogenic semigroup $T = \langle b \mid b^{m+1} = b\rangle$ for the integers $n, m \geq 3$, when it is not a group and even it is not a commutative semigroup. Indeed we prove that:

Proposition A. For integers $n, m \geq 3$ when m is even, the semidirect product of monogenic semigroup $S = \langle a \mid a^{n+1} = a\rangle$ by the monogenic semigroup $T = \langle b \mid b^{m+1} = b\rangle$ is a non-commutative semigroup of order mn and may be presented by the presentation:

$$\pi = \langle A, B \mid A^{n+1} = A, B^{m+1} = B, BA = A^{n-1}B, B^m = A^n\rangle.$$

Corollary B. For integers $n, m \geq 3$ when m is odd the semigroup presented by

$$\pi = \langle A, B \mid A^{n+1} = A, B^{m+1} = B, BA = A^{n-1}B, B^m = A^n\rangle$$

is a finite commutative semigroup of order $\frac{(3+(-1)^n)m}{2}$.

Corollary C. For every integers $n, m \geq 3$ the direct product of monogenic semigroups S and T may be presented by:

$$\pi_1 = \langle A, B \mid A^{n+1} = A, B^{m+1} = B, B^m = A^n, AB = BA \rangle.$$

2 Preliminaries

For two semigroups S and T (finite or infinite) first we recall the definition of semidirect product of S by T.

Definition 2.1 *If $\phi : T \to End(S)$ is a homomorphism then the set $S \times T$ is a semigroup under the multiplication defined by*

$$(s_1, t_1).(s_2, t_2) = (s_1 \phi_{t_1}(s_2), t_1 t_2)$$

where $\phi_{t_1} = \phi(t_1) \in End(S)$. This semigroups will be denoted by $S \times_\phi T$ and is called the semidirect product of S by T with respect to ϕ.

Obviously, when $\phi(t) = id_S$ for every $t \in T$ then the semidirect product will be reduced to the direct product of semigroups S and T.

Getting such homomorphism to establish a semidirect product is our main goal and we are going to do this for the monogenic semigroups $S = \langle a \mid a^{n+1} = a \rangle$ and $T = \langle b \mid b^{m+1} = b \rangle$. The key lemma to construct ϕ is the following:

Lemma 2.2 *For every integer $n \geq 3$ the semigroup S accepts an involution homomorphism.*

Proof. Define $\theta : S \to S$ by $\theta(a) = a^{n-1}$. Obviously, θ is a homomorphism and,

$$\theta^2(a) = a^{(n-1)^2} = a^{4+(n-3)(n+1)} = a^4 (a^{n+1})^{n-3} = a^4 a^{n-3} = a^{n+1} = a.$$

So, $\theta^2 = id_S$.

Lemma 2.3 *For the integers $n, m \geq 3$ where m is even, the mapping $\phi : T \to End(S)$ defined by*

$$\phi(t) = \begin{cases} \theta, & \text{if } t = b^i \text{ and } i \text{ is odd} \\ id_S, & \text{if } t = b^i \text{ and } i \text{ is even} \end{cases}$$

is a semigroup homomorphism.

Proof. For two arbitrary elements $x = b^i$ and $y = b^j$ of T we may consider four cases for i and j to show that $\phi(xy) = \phi(x)\phi(y)$ holds by considering the definition of θ and the Lemma 2.2.

Considering the semigroups $S = \langle a \mid a^{n+1} = a \rangle$ and $T = \langle b \mid b^{m+1} = b \rangle$ ($n, m \geq 3$ and m is even), we get the following result concerning the properties of certain special elements of the semidirect product $S \times_\phi T$.

Lemma 2.4 *The elements $A = (a, b^m)$ and $B = (a^n, b)$ of the semigroup $S \times_\phi T$ satisfy the relators $BA = A^{n-1}B$, $A^n = B^m$, $A^{n+1} = A$ and $B^{m+1} = B$.*

Proof. We use the results of above lemmas to prove the assertions and we only prove the first relator. The other relators may be proved in a similar way. Using the definition of the multiplication in $S \times_\phi T$ we get:

$$\begin{aligned} BA &= (a^n, b)(a, b^m), \\ &= (a^n\phi_b(a), bb^m), \\ &= (a^n\theta(a), b), \quad \text{(for, } b^{m+1} = b), \\ &= (a^n a^{n-1}, b), \\ &= (a^{n+1}a^{n-2}, b) \\ &= (a^{n-1}, b), \quad \text{(for, } a^{n+1} = a). \end{aligned}$$

On the other hand, since m is even, we may use an induction method to show that $A^k = (a^k, b^m)$, for every positive integer k. So,

$$\begin{aligned} A^{n-1}B &= (a^{n-1}, b^m)(a^n, b), \\ &= (a^{2n-1}, b), \quad \text{(for, } b^{m+1} = b), \\ &= (a^{(n+1)+(n-2)}, b), \\ &= (a^{n-1}, b). \end{aligned}$$

Consequently, $BA = A^{n-1}B$.

Lemma 2.5 *For every i and j where $1 \leq i \leq n-1$ and $1 \leq j \leq m-1$, the elements $A = (a, b^m)$ and $B = (a^n, b)$ of the semigroup $S \times_\phi T$ satisfy the relators $BA^i = A^{n-i}B$ and $BAB^j = A^{n-1}B^j$.*

Proof. The proofs are straight forward by using an induction method and using the relator $BA = A^{n-1}B$ of the above lemma.

3 The Proofs of main results

By using the results of previous section we are ready to prove the assertions of the Proposition A, Corollary B and the Corollary C.

The proof of Proposition A. Obviously, the semigroup $S \times_\phi T$ is of order nm and this a is non-commutative semigroup because of

$$AB = (a, b^m)(a^n, b) = (a, b) \neq (a^n, b)(a, b^m) = (a^{n+n-1}, b) = (a^{n-1}, b) = BA.$$

As usual, the semigroup presented by a presentation π will be denoted by $Sg(\pi)$ (see [1], for example). Now, to complete the proof we have to show that the semigroup $S \times_\phi T$ is isomorphic to $Sg(\pi)$ where,

$$\pi = \langle A, B \mid BA = A^{n-1}B, A^{n+1} = A, B^{m+1} = B, B^m = A^n \rangle.$$

By the relators of π we get that $Sg(\pi) = X \cup Y \cup Z$ where,

$$\begin{aligned} X &= \{A^i \mid 1 \leq i \leq n\}, \\ Y &= \{B^j \mid 1 \leq j \leq m-1\}, \\ Z &= \{A^iB^j \mid 1 \leq i \leq n-1,\ 1 \leq j \leq m-1\}. \end{aligned}$$

So, the semigroup $Sg(\pi)$ is of order $nm = n + m - 1 + (n-1)(m-1)$. This proves that tow semigroups $Sg(\pi)$ and $S \times_\phi T$ are of same order and both are non-commutative. It is now necessary to show that the semigroup $S \times_\phi T$ could be generated by tow elements $A = (a, b^m)$ and $B = (a^n, b)$.

Using the results of Lemmas 2.4 and 2.5 gives us:

$$(a^i, b^j) = \begin{cases} B^{j-1}A^{n-i}B, & 1 \leq i \leq n-1 \textit{ and } 1 \leq j \leq m-1, \\ A^i, & 1 \leq i \leq n-1 \textit{ and } j = m, \\ B^j, & \text{i} = n \textit{ and } 1 \leq j \leq m-1, \\ A^n (= B^m), & \text{i} = n \textit{ and } j = m. \end{cases}$$

This completes the proof.

Proving the Corollary B needs the following lemma detecting certain new relators by manipulating the existing relators of $Sg(\pi)$, for odd values of m.

Lemma 3.1 *Let $n, m \geq 3$ and m is odd. Then the relator $A = A^{n-1}$ holds in the semigroup $Sg(\pi)$ where,*

$$\pi = \langle A, B \mid BA = A^{n-1}B, A^{n+1} = A, B^{m+1} = B, B^m = A^n \rangle.$$

Proof. The relator $BA = A^{n-1}B$ yields:

$$B^2A = BA^{n-1}B = A^{n-1}BA^{n-2}B = \ldots = (A^{n-1})^{n-1}B^2 = A^{(n-1)^2}B^2.$$

Then, in a similar way we get:

$$B^3A = A^{(n-1)^3}B^3, B^4A = A^{(n-1)^4}B^4, \ldots, B^mA = A^{(n-1)^m}B^m.$$

So, by using the relator $A^n = b^m$ we get $A^{n+1} = A^{n+(n-1)^m}$. This relator in turn yields $A = A^{(n-1)^m}$. To prove the assertion we have to show that $A^{n-1} = A^{(n-1)^m}$, and this is true because of the following computations:

$$\begin{aligned}
&A^{(n-1)^2} = A^{(n+1)^2-4(n+1)+4} = A^{1-4+4} = A,\\
&A^{(n-1)^3} = (A^{(n-1)^2})^{(n-1)} = A^{n-1},\\
&A^{(n-1)^4} = (A^{(n-1)^3})^{(n-1)} = A^{(n-1)^2} = A,\\
&\vdots\\
&A^{(n-1)^m} = A^{n-1}, \text{ (for, } m \text{ is odd).}
\end{aligned}$$

The proof of Corollary B. By the result of Lemma 3.1 the relator $BA = A^{n-1}B$ becomes to $BA = AB$, showing that $Sg(\pi)$ is commutative. Also, the relators $A^{n+1} = A$ and $A^{n-1} = A$ yield the relator $A^3 = A$. Now we may consider two cases for n.

Case 1: n is even. In this case $Sg(\pi)$ may be described as the following disjoint union of sets:

$$Sg(\pi) = \{A, A^2\} \cup \{B, B^2, \dots, B^{m-1}\} \cup \{AB^i \mid 1 \leq i \leq m-1\},$$

for, $A^4 = A^2, A^5 = A, A^6 = A^2, \dots, A^n = A^2$ and $B^m = A^2$. Consequently, $\mid Sg(\pi) \mid = 2m$.

Case 2: n is odd. As in the last case we get $A^n = A^2$. However, $A^{n+1} = A^3 = A$ yields $A^2 = A^3 = \dots = A^n = A$. Showing that

$$Sg(\pi) = \{A\} \cup \{B, B^2, \dots, B^{m-1}\}.$$

Hence, $\mid Sg(\pi) \mid = m$.

Combining two cases we get the result $\mid Sg(\pi) \mid = \frac{(3+(-1)^n)m}{2}$.

The proof of Corollary C. The mapping $\phi : T \to End(S)$ defined by $\phi(t) = id_S$ gives rise to the multiplication $(s,t)(s',t') = (ss',tt')$ on $S \times T$, for all values of $m, n \geq 3$. By the same way as in the proof of Proposition A we may get a desired presentation for the semigroup $S \times T$. Indeed, by this multiplication the generators $A = (a, b^m)$ and $B = (a^n, b)$ generate the semigroup $S \times T$ (for, $(a^i, b^j) = A^iB^j$ holds for all $1 \leq i \leq n$ and $1 \leq j \leq m$) and satisfying the relators $A^{n+1} = A, B^{m+1} = B, B^m = A^n, AB = BA$, i.e.; $S \times T$ is isomorphic to $Sg(\pi_1)$.

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Received: July 15, 2013