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The Best Possible Lehmer Mean Bounds for a Convex Combination of Logarithmic and Harmonic Means[1]

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Abstract

For $r \in \mathbb{R}$ and $a, b > 0$ the Lehmer mean $L_r(a,b)$, logarithmic mean $L(a,b)$, and harmonic mean $H(a,b)$ are defined by

$$L_r(a,b) = \frac{a^{r+1}+b^{r+1}}{a^r+b^r}, \quad L(a,b) = \begin{cases} \frac{b-a}{\log b - \log a}, & b \neq a, \\ a, & b = a, \end{cases}$$

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and $H(a,b)=\frac{2ab}{a+b}$, respectively. In this paper, we answer the question: For $\alpha\in(0,1)$, what are the largest value p and least value q such that the double inequality $L_q(a,b)>\alpha H(a,b)+(1-\alpha)L(a,b)>L_p(a,b)$ holds for all $a,b>0$ with $a\neq b$?

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1. Introduction

For $r\in\mathbb{R}$ and $a,b>0$ the Lehmer mean $L_r(a,b)$, logarithmic mean $L(a,b)$, and harmonic mean $H(a,b)$ are defined by

$$L_r(a,b)=\frac{a^{r+1}+b^{r+1}}{a^r+b^r}, \tag{1.1}$$

$$L(a,b)=\begin{cases}\frac{b-a}{\log b-\log a}, & b\neq a,\\ a, & b=a,\end{cases} \tag{1.2}$$

and

$$H(a,b)=\frac{2ab}{a+b}. \tag{1.3}$$

It is well-known that $L_r(a,b)$ is continuous and strictly increasing with respect to $r\in\mathbb{R}$ for fixed a and b with $a\neq b$. In the recent past, the Lehmer mean has been attracted the attention of many mathematicians [1-14]. Many means are the special cases of the Lehmer mean, for example,

$$A(a,b)=\frac{a+b}{2}=L_0(a,b) \text{ is the arithmetic mean,}$$

$$G(a,b)=\sqrt{ab}=L_{-\frac{1}{2}}(a,b) \text{ is the geometric mean,}$$

$$H(a,b)=\frac{2ab}{a+b}=L_{-1}(a,b) \text{ is the harmonic mean.}$$

Recently, the logarithmic mean has been the subject of intensive research. In particular, many remarkable inequalities for the logarithmic mean can be found in literatures [15-30]. It might be surprising that the logarithmic mean has applications in physics [31], economics [32], and even in meteorology [33]. In [31], the authors study a variant of Jensen's function involving logarithmic mean, which appears in a heat conduction problem. For $p\in\mathbb{R}$, let

$$M_p(a,b)=\begin{cases}(\frac{a^p+b^p}{2})^{\frac{1}{p}}, & p\neq 0,\\ \sqrt{ab}, & p=0\end{cases} \quad\text{and}\quad I(a,b)=\begin{cases}\frac{1}{e}(\frac{b^b}{a^a})^{\frac{1}{b-a}}, & b\neq a,\\ a, & b=a\end{cases}$$

be the pth power mean and identric mean of two positive real numbers a and b, respectively. Then it is well known that

$$\begin{aligned}&\min\{a,b\} < H(a,b) = L_{-1}(a,b) = M_{-1}(a,b) < G(a,b)\\ &= L_{-\frac{1}{2}}(a,b) = M_0(a,b) < L(a,b) < I(a,b) < A(a,b)\\ &= L_0(a,b) = M_1(a,b) < \max\{a,b\}\end{aligned} \tag{1.4}$$

for all $a, b > 0$ with $a \neq b$.

In [13], Alzer presented the following sharp upper and lower Lehmer mean bounds for identric mean $I(a,b)$:

$$L_{-\frac{1}{6}}(a,b) < I(a,b) < L_0(a,b)$$

for all $a, b > 0$ with $a \neq b$.

Stolarsky [12] established that

$$M_{2n+1}(a,b) \leq L_n(a,b)$$

for all $a, b > 0$ with $a \neq b$.

In [24, 34, 35], the authors presented bounds for L and I in terms of G and A as follows:

$$G^{\frac{2}{3}}(a,b)A^{\frac{1}{3}}(a,b) < L(a,b) < \frac{2}{3}G(a,b) + \frac{1}{3}A(a,b)$$

and

$$I(a,b) > \frac{1}{3}G(a,b) + \frac{2}{3}A(a,b)$$

for all $a, b > 0$ with $a \neq b$.

The following inequalities can be found in [26]:

$$G^{\frac{1}{2}}(a,b)A^{\frac{1}{2}}(a,b) < L^{\frac{1}{2}}(a,b)I^{\frac{1}{2}}(a,b) < \frac{1}{2}(L(a,b)+I(a,b)) < \frac{1}{2}(G(a,b)+A(a,b)).$$

The following sharp bounds for L, I, $(IL)^{\frac{1}{2}}$, and $\frac{1}{2}(I+L)$ in terms of power means are proved in [25-27, 36-39]:

$$M_0(a,b) < L(a,b) < M_{\frac{1}{3}}(a,b), \quad M_{\frac{2}{3}}(a,b) < I(a,b) < M_{\log 2}(a,b),$$

$$M_0(a,b) < \sqrt{I(a,b)L(a,b)} < M_{\frac{1}{2}}(a,b) \quad \text{and} \quad \frac{1}{2}(I(a,b) + L(a,b)) < M_{\frac{1}{2}}(a,b)$$

for all $a, b > 0$ with $a \neq b$, and the given parameters are best possible.

Alzer and Qiu [30] proved that $M_c(a,b) < \frac{1}{2}(L(a,b)+I(a,b))$ for all $a, b > 0$ with $a \neq b$ if and only if $c \leq \log 2/(1 + \log 2)$.

The purpose of this paper is to answer the question: For $\alpha \in (0,1)$, what are the largest value p and least value q such that the double inequality $L_q(a,b) >$

$\alpha H(a,b) + (1-\alpha)L(a,b) > L_p(a,b)$ holds for all $a, b > 0$ with $a \neq b$.

2. Lemmas

In order to prove our main result, we need several lemmas which we present in this section.

Lemma 2.1. Suppose that $\alpha \in (0,1)$ and $x \in (1,\infty)$. If $p = -\frac{2\alpha+1}{3}$, then

$$x^{p+2} + (1-2\alpha)x^{p+1} + (1-2\alpha)x + 1 > 0. \tag{2.1}$$

Proof. Let $h(x) = x^{p+2}+(1-2\alpha)x^{p+1}+(1-2\alpha)x+1$, then simple computations lead to

$$\lim_{x\to 1} h(x) = 4(1-\alpha) > 0, \tag{2.2}$$

$$h'(x) = (p+2)x^{p+1} + (1-2\alpha)(p+1)x^p + 1 - 2\alpha, \tag{2.3}$$

$$\lim_{x\to 1} h'(x) = 2(p+2)(1-\alpha) > 0, \tag{2.4}$$

$$h''(x) = (p+1)x^{p-1}h_1(x), \tag{2.5}$$

where $h_1(x) = (p+2)x + (1-2\alpha)p$.

Note that

$$\lim_{x\to 1} h_1(x) = \alpha^2 + 1 + \frac{1}{3}(\alpha-1)^2 > 0, \tag{2.6}$$

$$h_1'(x) = p + 2 > 0. \tag{2.7}$$

Therefore, inequality (2.1) follows from (2.2)-(2.7). □

Lemma 2.2. If $\alpha \in (0,1)$, then $736\alpha^6 + 1064\alpha^5 - 908\alpha^4 + 2338\alpha^3 - 5111\alpha^2 + 3665\alpha + 565 > 0$.

Proof. We clearly see that

$$\begin{aligned} & 736\alpha^6 + 1064\alpha^5 - 908\alpha^4 + 2338\alpha^3 - 5111\alpha^2 + 3665\alpha + 565 \\ = \; & \alpha^2(736\alpha^4 - 908\alpha^2 + 281) + 1064\alpha^5 + 2338\alpha^3 - 5392\alpha^2 \\ & +3665\alpha + 565 \\ > \; & \alpha^2(736\alpha^4 - 908\alpha^2 + 281) + 2338\alpha^3 - 5392\alpha^2 + 3665\alpha + 565\alpha \\ = \; & \alpha^2(736\alpha^4 - 908\alpha^2 + 281) + \alpha(2338\alpha^2 - 5392\alpha + 4230). \end{aligned} \tag{2.8}$$

The discriminants Δ_1 and Δ_2 of the quadratic functions $736t^2 - 908t + 281$ and $2338t^2 - 5392t + 4320$ satisfy

$$\Delta_1 = (-908)^2 - 4 \times 736 \times 281 = -2800 < 0 \tag{2.9}$$

and

$$\Delta_2 = (-5392)^2 - 4 \times 2338 \times 4230 = -10485296 < 0 \tag{2.10}$$

respectively. Therefore, Lemma 2.2 follows from (2.8)-(2.10). □

Lemma 2.3. If $\alpha \in (0,1)$, then $808\alpha^5 + 2836\alpha^4 + 1750\alpha^3 + 1343\alpha^2 - 3124\alpha + 1085 > 0$.

Proof. We clearly see that

$$\begin{aligned} & 808\alpha^5 + 2836\alpha^4 + 1750\alpha^3 + 1343\alpha^2 - 3124\alpha + 1085 \\ > \ & 1750\alpha^3 + 1350\alpha^2 - 3150\alpha + 1050 \\ = \ & 50(35\alpha^3 + 27\alpha^2 - 63\alpha + 21). \end{aligned} \tag{2.11}$$

It is not difficult to verify that

$$\begin{aligned} & \min_{\alpha\in(0,1)}(35\alpha^3 + 27\alpha^2 - 63\alpha + 21) \\ = \ & 35(\tfrac{4\sqrt{51}-9}{35})^3 + 27(\tfrac{4\sqrt{51}-9}{35})^2 - 63(\tfrac{4\sqrt{51}-9}{35}) + 21 \\ = \ & 0.333\ldots > 0. \end{aligned} \tag{2.12}$$

Therefore, Lemma 2.3 follows from (2.11) and (2.12). □

Lemma 2.4. If $\alpha \in (0,1)$, then $304\alpha^5 + 724\alpha^4 - 488\alpha^3 + 839\alpha^2 - 3361\alpha + 2387 > 0$.

Proof. We clearly see that

$$\begin{aligned} & 304\alpha^5 + 724\alpha^4 - 488\alpha^3 + 839\alpha^2 - 3361\alpha + 2387 \\ > \ & 724\alpha^4 - 488\alpha^3 + 488\alpha^2 + 351\alpha^2 - 3361\alpha + 2387 \\ > \ & 724\alpha^4 + 351\alpha^4 - 3361\alpha + 2387 \\ = \ & 1075\alpha^4 - 3361\alpha + 2387. \end{aligned} \tag{2.13}$$

It is easy to verify that

$$\begin{aligned} & \min_{\alpha\in(0,1)}(1075\alpha^4 - 3361\alpha + 2387) \\ = \ & 1075(\sqrt[3]{\tfrac{3361}{4300}})^4 - 3361(\sqrt[3]{\tfrac{3361}{4300}}) + 2387 \\ = \ & 64.995\ldots > 0. \end{aligned} \tag{2.14}$$

Therefore, Lemma 2.4 follows from (2.13) and (2.14). □

3. Main Result

Theorem 3.1. If $\alpha \in (0,1)$, then $L_0(a,b) > \alpha H(a,b) + (1-\alpha)L(a,b) > L_{-\frac{2\alpha+1}{3}}(a,b)$ for all $a, b > 0$ with $a \neq b$, and $L_0(a,b)$ and $L_{-\frac{2\alpha+1}{3}}(a,b)$ are the best possible upper and lower Lehmer mean bounds for the sum $\alpha H(a,b) + (1-\alpha)L(a,b)$.

Proof. Without loss of generality, we assume that $a > b$. From (1.4) we clearly see that $L_0(a,b) > \alpha H(a,b) + (1-\alpha)L(a,b)$.

Next, we prove that

$$\alpha H(a,b) + (1-\alpha)L(a,b) > L_{-\frac{2\alpha+1}{3}}(a,b). \tag{3.1}$$

Let $x = \frac{a}{b} > 1$ and $p = -\frac{2\alpha+1}{3}$, then (1.1)-(1.3) lead to

$$\begin{aligned} &\alpha H(a,b) + (1-\alpha)L(a,b) - L_p(a,b) \\ = \ & \frac{b[x^{p+2} + (1-2\alpha)x^{p+1} + (1-2\alpha)x + 1]}{(1+x)(1+x^p)\log x} f(x), \end{aligned} \tag{3.2}$$

where $f(x) = \frac{(1-\alpha)(x^{p+2} - x^p + x^2 - 1)}{x^{p+2} + (1-2\alpha)x^{p+1} + (1-2\alpha)x + 1} - \log x$. Note that

$$\lim_{x\to 1} f(x) = 0, \tag{3.3}$$

$$f'(x) = \frac{f_1(x)}{x[x^{p+2} + (1-2\alpha)x^{p+1} + (1-2\alpha)x + 1]^2}, \tag{3.4}$$

where $f_1(x) = -x^{2p+4} + (2\alpha^2 + \alpha - 1)x^{2p+3} - (4\alpha^2 - 2\alpha - 1)x^{2p+2} + (2\alpha^2 - 3\alpha + 1)x^{2p+1} - p(1-\alpha)x^{p+4} + 2\alpha(2\alpha - 1)x^{p+3} - 2(4\alpha^2 - 2\alpha + \alpha p - p)x^{p+2} + 2\alpha(2\alpha - 1)x^{p+1} - p(1-\alpha)x^p + (2\alpha^2 - 3\alpha + 1)x^3 - (4\alpha^2 - 2\alpha - 1)x^2 + (2\alpha^2 + \alpha - 1)x - 1$,

$$\lim_{x\to 1} f_1(x) = 0. \tag{3.5}$$

Let $f_2(x) = x^{4-p} f_1^{(4)}(x)$, $f_3(x) = \frac{1}{2(p+1)} f_2'(x)$, $f_4(x) = \frac{1}{2} f_3'(x)$, $f_5(x) = x^{4-p} f_4'''(x)$, $f_6(x) = \frac{1}{(p+1)(p+2)} f_5'(x)$, and $f_7(x) = \frac{1}{2(p+3)(2p+3)} f_6'(x)$, then simple computations yield that

$$\begin{aligned} f_1'(x) = \ & -2(p+2)x^{2p+3} + (2p+3)(2\alpha^2 + \alpha - 1)x^{2p+2} - 2(p+1) \\ & \times(4\alpha^2 - 2\alpha - 1)x^{2p+1} + (2p+1)(2\alpha^2 - 3\alpha + 1)x^{2p} \\ & -p(p+4)(1-\alpha)x^{p+3} + 2\alpha(p+3)(2\alpha - 1)x^{p+2} - 2(p+2) \\ & \times(4\alpha^2 - 2\alpha + \alpha p - p)x^{p+1} + 2\alpha(p+1)(2\alpha - 1)x^p \\ & -p^2(1-\alpha)x^{p-1} + 3(2\alpha^2 - 3\alpha + 1)x^2 - 2(4\alpha^2 - 2\alpha - 1)x \\ & +2\alpha^2 + \alpha - 1, \end{aligned}$$

$$\lim_{x\to 1} f_1'(x) = 0, \tag{3.6}$$

$$\begin{aligned} f_1''(x) = \ & -2(p+2)(2p+3)x^{2p+2} + 2(p+1)(2p+3)(2\alpha^2 + \alpha - 1)x^{2p+1} \\ & -2(p+1)(2p+1)(4\alpha^2 - 2\alpha - 1)x^{2p} + 2p(2p+1) \\ & \times(2\alpha^2 - 3\alpha + 1)x^{2p-1} - p(p+3)(p+4)(1-\alpha)x^{p+2} \\ & +2\alpha(p+2)(p+3)(2\alpha - 1)x^{p+1} - 2(p+1)(p+2) \\ & \times(4\alpha^2 - 2\alpha + \alpha p - p)x^p + 2\alpha p(p+1)(2\alpha - 1)x^{p-1} \\ & -p^2(p-1)(1-\alpha)x^{p-2} + 6(2\alpha^2 - 3\alpha + 1)x \\ & -2(4\alpha^2 - 2\alpha - 1), \end{aligned}$$

$$\lim_{x\to 1} f_1''(x) = 0, \tag{3.7}$$

$$\begin{aligned} f_1'''(x) = \; & -4(p+1)(p+2)(2p+3)x^{2p+1} + 2(p+1)(2p+1)(2p+3) \\ & \times(2\alpha^2+\alpha-1)x^{2p} - 4p(p+1)(2p+1)(4\alpha^2-2\alpha-1)x^{2p-1} \\ & +2p(2p-1)(2p+1)(2\alpha^2-3\alpha+1)x^{2p-2} - p(p+2)(p+3) \\ & \times(p+4)(1-\alpha)x^{p+1} + 2\alpha(p+1)(p+2)(p+3)(2\alpha-1)x^p \\ & -2p(p+1)(p+2)(4\alpha^2-2\alpha+\alpha p-p)x^{p-1} + 2\alpha p(p-1) \\ & \times(p+1)(2\alpha-1)x^{p-2} - p^2(p-1)(p-2)(1-\alpha)x^{p-3} \\ & +6(2\alpha^2-3\alpha+1), \end{aligned}$$

$$\lim_{x\to 1} f_1'''(x) = 0, \tag{3.8}$$

$$\begin{aligned} f_2(x) = \; & -4(p+1)(p+2)(2p+1)(2p+3)x^{p+4} + 4p(p+1)(2p+1) \\ & \times(2p+3)(2\alpha^2+\alpha-1)x^{p+3} - 4p(p+1)(2p-1)(2p+1) \\ & \times(4\alpha^2-2\alpha-1)x^{p+2} + 4p(p-1)(2p-1)(2p+1) \\ & \times(2\alpha^2-3\alpha+1)x^{p+1} - p(p+1)(p+2)(p+3)(p+4)(1-\alpha)x^4 \\ & +2\alpha p(p+1)(p+2)(p+3)(2\alpha-1)x^3 - 2p(p-1)(p+1)(p+2) \\ & \times(4\alpha^2-2\alpha+\alpha p-p)x^2 + 2\alpha p(p-1)(p-2)(p+1)(2\alpha-1)x \\ & -p^2(p-1)(p-2)(p-3)(1-\alpha), \end{aligned}$$

$$\lim_{x\to 1} f_2(x) = \frac{32(1-\alpha)^2(10\alpha^2+25\alpha+1)}{27} > 0, \tag{3.9}$$

$$\begin{aligned} f_3(x) = \; & -2(p+2)(p+4)(2p+1)(2p+3)x^{p+3} + 2p(p+3)(2p+1) \\ & \times(2p+3)(2\alpha^2+\alpha-1)x^{p+2} - 2p(p+2)(2p-1)(2p+1) \\ & \times(4\alpha^2-2\alpha-1)x^{p+1} + 2p(p-1)(2p-1)(2p+1) \\ & \times(2\alpha^2-3\alpha+1)x^p - 2p(p+2)(p+3)(p+4)(1-\alpha)x^3 \\ & +3\alpha p(p+2)(p+3)(2\alpha-1)x^2 - 2p(p-1)(p+2) \\ & \times(4\alpha^2-2\alpha+\alpha p-p)x + \alpha p(p-1)(p-2)(2\alpha-1), \end{aligned}$$

$$\lim_{x\to 1} f_3(x) = \frac{64(1-\alpha)^2(10\alpha^2+25\alpha+1)}{27} > 0, \tag{3.10}$$

$$\begin{aligned} f_4(x) = \; & -(p+2)(p+3)(p+4)(2p+1)(2p+3)x^{p+2} + p(p+2)(p+3) \\ & \times(2p+1)(2p+3)(2\alpha^2+\alpha-1)x^{p+1} - p(p+1)(p+2)(2p-1) \\ & \times(2p+1)(4\alpha^2-2\alpha-1)x^p + p^2(p-1)(2p-1)(2p+1) \\ & \times(2\alpha^2-3\alpha+1)x^{p-1} - 3p(p+2)(p+3)(p+4)(1-\alpha)x^2 \\ & +3\alpha p(p+2)(p+3)(2\alpha-1)x - p(p-1)(p+2) \\ & \times(4\alpha^2-2\alpha+\alpha p-p), \end{aligned}$$

$$\lim_{x\to 1} f_4(x) = \frac{2}{27}(1-\alpha)[248\alpha^4+148\alpha^3+786\alpha(1-\alpha)+241\alpha+65] > 0, \tag{3.11}$$

$$\begin{aligned} f_4'(x) = \; & -(p+2)^2(p+3)(p+4)(2p+1)(2p+3)x^{p+1} + p(p+1)(p+2) \\ & \times(p+3)(2p+1)(2p+3)(2\alpha^2+\alpha-1)x^p - p^2(p+1)(p+2) \\ & \times(2p-1)(2p+1)(4\alpha^2-2\alpha-1)x^{p-1} + p^2(p-1)^2(2p-1) \\ & \times(2p+1)(2\alpha^2-3\alpha+1)x^{p-2} - 6p(p+2)(p+3)(p+4) \\ & \times(1-\alpha)x + 3\alpha p(p+2)(p+3)(2\alpha-1), \end{aligned}$$

$$\begin{aligned}\lim_{x\to 1} f_4'(x) &= \frac{2}{243}(1-\alpha)[\alpha(1-\alpha)(2752\alpha^3+4040\alpha^2+14978) \\ &\quad +2748\alpha^3+2105\alpha+1465] \\ &> 0,\end{aligned} \tag{3.12}$$

$$\begin{aligned}f_4''(x) = & -(p+1)(p+2)^2(p+3)(p+4)(2p+1)(2p+3)x^p+p^2(p+1) \\ & \times(p+2)(p+3)(2p+1)(2p+3)(2\alpha^2+\alpha-1)x^{p-1}-p^2(p-1) \\ & \times(p+1)(p+2)(2p-1)(2p+1)(4\alpha^2-2\alpha-1)x^{p-2}+p^2 \\ & \times(p-1)^2(p-2)(2p-1)(2p+1)(2\alpha^2-3\alpha+1)x^{p-3} \\ & -6p(p+2)(p+3)(p+4)(1-\alpha),\end{aligned}$$

$$\begin{aligned}\lim_{x\to 1} f_4''(x) = & \frac{8}{243}(1-\alpha)(736\alpha^6+1064\alpha^5-908\alpha^4+2338\alpha^3 \\ & -5111\alpha^2+3665\alpha+565),\end{aligned} \tag{3.13}$$

$$\lim_{x\to+\infty} f_4''(x) = -6p(p+2)(p+3)(p+4)(1-\alpha) > 0 \tag{3.14}$$

$$\begin{aligned}f_5(x) = & -p(p+1)(p+2)^2(p+3)(p+4)(2p+1)(2p+3)x^3+p^2(p-1) \\ & \times(p+1)(p+2)(p+3)(2p+1)(2p+3)(2\alpha^2+\alpha-1)x^2-p^2 \\ & \times(p-1)(p-2)(p+1)(p+2)(2p-1)(2p+1)(4\alpha^2-2\alpha-1)x \\ & +p^2(p-1)^2(p-2)(p-3)(2p-1)(2p+1)(2\alpha^2-3\alpha+1),\end{aligned}$$

$$\begin{aligned}\lim_{x\to 1} f_5(x) &= \frac{8}{2187}(1-\alpha)(1-4\alpha)(2\alpha+1)[808\alpha^5+2836\alpha^4 \\ &\quad +1750\alpha^3+1343\alpha^2-3124\alpha+1781\alpha+1085]\end{aligned} \tag{3.15}$$

$$\begin{aligned}f_6(x) = & -3p(p+2)(p+3)(p+4)(2p+1)(2p+3)x^2+2p^2(p-1) \\ & \times(p+3)(2p+1)(2p+3)(2\alpha^2+\alpha-1)x-p^2(p-1) \\ & \times(p-2)(2p-1)(2p+1)(4\alpha^2-2\alpha-1),\end{aligned}$$

$$\begin{aligned}\lim_{x\to 1} f_6(x) = & \frac{4}{729}(1-4\alpha)(2\alpha+1)(304\alpha^5+724\alpha^4-488\alpha^3 \\ & +839\alpha^2-3361\alpha+2387)\end{aligned} \tag{3.16}$$

$$f_7(x) = -3p(p+2)(p+4)(2p+1)x+p^2(p-1)(2p+1)(2\alpha^2+\alpha-1)$$

$$\lim_{x\to 1} f_7(x) = \frac{1}{81}(1-4\alpha)(2\alpha+1)(169-8\alpha^4-24\alpha^3-2\alpha^2-90\alpha), \tag{3.17}$$

$$f_7'(x) = -(1-4\alpha)p(p+2)(p+4). \tag{3.18}$$

We divide the proof of inequality (3.1) into two cases.

Case 1. If $\alpha\in(0,\frac{1}{4}]$, then from Lemmas 2.2-2.4 and (3.13) together with (3.15)-(3.18) we clearly see that

$$\lim_{x\to 1} f_4''(x) > 0, \tag{3.19}$$

$$\lim_{x\to 1} f_5(x) > 0, \tag{3.20}$$

$$\lim_{x\to 1} f_6(x) > 0, \tag{3.21}$$

$$\lim_{x\to 1} f_7(x) > 0, \tag{3.22}$$

$$f_7{}'(x) > 0. \tag{3.23}$$

It easily follows from (3.3)-(3.12) and (3.19)-(3.23) that

$$f(x) > 0 \tag{3.24}$$

for $x > 1$.

Therefore inequality (3.1) follows from (3.2) and (3.24) together with Lemma 2.1.

Case 2. If $\alpha \in (\frac{1}{4}, 1)$, then from Lemmas 2.2-2.4 and (3.13) together with (3.15)-(3.18) we clearly see that (3.19) again holds and

$$\lim_{x\to 1} f_5(x) < 0, \tag{3.25}$$

$$\lim_{x\to 1} f_6(x) < 0, \tag{3.26}$$

$$\lim_{x\to 1} f_7(x) < 0, \tag{3.27}$$

$$f_7{}'(x) < 0. \tag{3.28}$$

Inequalities (3.25)-(3.28) imply that $f_4''(x)$ is strictly decreasing in $(1,\infty)$. Then from (3.14) and (3.19) together with the monotonicity of $f_4''(x)$ we know that

$$f_4''(x) > 0 \tag{3.29}$$

for $x \in (1, \infty)$.

Therefore, inequality (3.1) follows from (3.2)-(3.12) and (3.29) together with Lemma 2.1.

Finally, we prove that $L_0(a,b)$ and $L_{-\frac{2\alpha+1}{3}}(a,b)$ are the best possible upper and lower Lehmer mean bounds for the sum $\alpha H(a,b) + (1-\alpha)L(a,b)$, respectively.

For any $\varepsilon > 0$ and $x > 0$, we have

$$\begin{aligned} & L_{-\frac{2\alpha+1}{3}+\varepsilon}((1+x)^3, 1) - [\alpha H((1+x)^3, 1) + (1-\alpha)L((1+x)^3, 1)] \\ = & \frac{J(x)}{3[1+(1+x)^{3\varepsilon-1-2\alpha}][1+(1+x)^3]\log(1+x)} \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} & \lim_{x\to+\infty} [\alpha H(x,1) + (1-\alpha)L(x,1) - L_{-\varepsilon}(x,1)] \\ = & \lim_{x\to+\infty} [\frac{2\alpha x}{1+x} + \frac{(1-\alpha)(x-1)}{\log x} - \frac{x^{1-\varepsilon}+1}{x^{-\varepsilon}+1}] \\ = & +\infty, \end{aligned} \tag{3.31}$$

where $J(x) = 3[1+(1+x)^{2+3\varepsilon-2\alpha}][1+(1+x)^3]\log(1+x) - 6\alpha(1+x)^3[1+(1+x)^{3\varepsilon-1-2\alpha}]\log(1+x) - (1-\alpha)[(1+x)^6-1][1+(1+x)^{3\varepsilon-1-2\alpha}]$.

Let $x \to 0$, making use of the Taylor expansion, one has

$$\begin{aligned} J(x) &= 3x[2+(2+3\varepsilon-2\alpha)x+\frac{1}{2}(2+3\varepsilon-2\alpha)(1+3\varepsilon-2\alpha)x^2 \\ &\quad +o(x^2)][2+3x+3x^2+o(x^2)][1-\frac{1}{2}x+\frac{1}{3}x^2+o(x^2)] \\ &\quad -6\alpha x[1+3x+3x^2+o(x^2)][2+(3\varepsilon-1-2\alpha)x+\frac{1}{2} \\ &\quad \times(3\varepsilon-2\alpha-1)(3\varepsilon-2\alpha-2)x^2+o(x^2)][1-\frac{1}{2}x+\frac{1}{3}x^2 \\ &\quad +o(x^2)]-(1-\alpha)x[6+15x+20x^2+o(x^2)][2+(3\varepsilon-1 \\ &\quad -2\alpha)x+\frac{1}{2}(3\varepsilon-2\alpha-1)(3\varepsilon-2\alpha-2)x^2+o(x^2)] \\ &= 27\varepsilon x^3+o(x^3). \end{aligned} \tag{3.32}$$

Equations (3.30)-(3.32) imply that for any $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon) > 0$ and $X = X(\varepsilon) > 1$ such that $\alpha H((1+x)^3, 1) + (1-\alpha)L((1+x)^3, 1) < L_{-\frac{2\alpha+1}{3}+\varepsilon}((1+x)^3, 1)$ for $x \in (0, \delta)$ and $L_{-\varepsilon}(x, 1) < \alpha H(x, 1) + (1-\alpha)L(x, 1)$ for $x \in (X, \infty)$. □

References

[1] M. K. Wang, Y. M. Chu and G. D. Wang, A sharp double inequality between the Lehmer and arithmetic-geometric means, Pac. J. Appl. Math., **4**(2012), no. 1, 1-25.

[2] Y. M. Chu, M. K. Wang and Y. F. Qiu, Optimal Lehmer mean bounds for the geometric and arithmetic combinations of arithmetic and Seiffert means, Azerb. J. Math., **2**(2012), no. 1, 3-9.

[3] Y. M. Chu and M. K. Wang, Optimal Lehmer mean bounds for the Toader mean, Results Math., **61**(2012), no. 3-4, 223-229.

[4] M. K. Wang, Y. M. Chu and G. D. Wang, A sharp double inequality between the Lehmer and arithmetic-geometric means, Pac. J. Appl. Math., **3**(2011), no. 4, 281-286.

[5] Y. F. Qiu, M. K. Wang, Y. M. Chu and G. D. Wang, Two sharp inequalities for Lehmer mean, identric mean and logarithmic mean, J. Math. Inequal., **5**(2011), no. 3, 301-306.

[6] M. K. Wang, Y. F. Qiu and Y. M. Chu, Sharp bounds for Seiffert means in terms of Lehmer means, J. Math. Inequal., **4**(2010), no. 4, 581-586.

[7] W. F. Xia and Y. M. Chu, The Schur harmonic convexity of Lehmer means, Int. Math. Forum, **4**(2009), no. 41-44, 2009-2015.

[8] S. Toader and G. Toader, Complementaries of Greek means with respect to Lehmer means, Automat. Comput. Appl. Math., **15**(2006), no. 2, 315-320.

[9] I. Costin and G. Toader, Generalized inverses of Lehmer means, Automat. Comput. Appl. Math., **14**(2005), no. 1, 111-117.

[10] S. R. Wassell, Rediscovering a family of means, Math. Intelligencer, **24**(2002), no. 2, 58-65.

[11] Zh. Liu, Remark on inequalities between Hölder and Lehmer means, J. Math. Anal. Appl., **247**(2000), no. 1, 309-313.

[12] K. B. Stolarsky, Hölder means, Lehmer means, and $x^{-1}\log\cosh x$, J. Math. Anal. Appl., **202**(1996), no. 3, 810-818.

[13] H. Alzer, Bestmögliche Abschätzungen für spezielle Mittelwerte, Zb. Rad. Prirod. -Mat. Fak. Ser. Mat., **23**(1993), no. 1, 331-346.

[14] H. Alzer, Über Lehmers Mittelwertfamilie, Elem. Math., **43**(1988), no. 2, 50-54.

[15] H. N. Hu, S. S. Wang and Y. M. Chu, Optimal upper power mean bound for the convex combination of Harmonic and logarithmic means, Pac. J. Appl. Math., **4**(2012), no. 1, 35-44.

[16] Y. M. Li, B. Y. Long and Y. M. Chu, Sharp bounds for the Neuman-Sándor mean in terms of generalized logarithmic mean, J. Math. Inequal., **6**(2012), no. 4, 567-577.

[17] Y. M. Chu, M. K. Wang and Z. K. Wang, Best possible inequalities among harmonic, geometric, logarithmic and Seiffert means, Math. Inequal. Appl., **15**(2012), no. 2, 412-422.

[18] Y. M. Chu, S. W. Shou and W. M. Gong, Inequalities between logarithmic, harmonic, arithmetic and centroidal means, J. Math. Appl., **2**(2011), no. 2, 1-5.

[19] Y. M. Chu and M. K. Wang, Optimal inequalities between harmonic, geometric, logarithmic, and arithmetic-geometric means, J. Appl. Math., **2011**(2011), Article ID 618929, 9 pages.

[20] Y. M. Chu, S. S. Wang and C. Zong, Optimal lower power mean bound for convex combination of harmonic and logarithmic means, Abstr. Appl. Anal., **2011**(2011), Article ID 520648, 9 pages.

[21] W. F. Xia and Y. M. Chu, Optimal inequalities related to the logarithmic, identric, arithmetic and harmonic means, Rev. Anal. Numér. Théor. Approx.,, **39**(2010), no. 2, 176-183.

[22] W. F. Xia, Y. M. Chu and G. D. Wang, The optimal upper and lower power mean bounds for a convex combination of the arithmetic and logarithmic means, Abstr. Appl. Anal., **2010**(2010), Article ID 604804, 9 pages.

[23] Y. M. Chu and W. F. Xia, Two optimal double inequalities between power mean and logarithmic mean, Comput. Math. Appl., **60**(2011), no. 1, 83-89.

[24] B. C. Carlson, The logarithmic mean, Amer. Math. Monthly, **79**(1972), 615-618.

[25] T. P. Lin, The power mean and the logarithmic mean, Amer. Math. Monthly, **81**(1974), 879-883.

[26] H. Alzer, Ungleichungen für Mittelwerte, Arch. Math., **47**(1986), no. 5, 422-426.

[27] F. Burk, Notes: The geometric, logarithmic, and arithmetic mean Inequality, Amer. Math. Monthly, **94**(1987), no. 6, 527-528.

[28] J. Sándor, On the identric and logarithmic means, Aequations Math., **40**(1990), no. 2-3, 261-270.

[29] G. Allasia, C. Giordano and J. Pečarić, On the arithmetic and logarithmic means with applications to stirling's formula, Atti Sem. Mat. Fis. Univ. Modena, **47**(1999), no. 2, 441-455.

[30] H. Alzer and S. L. Qiu, Inequalities for means in two variables, Arch. Math., **80**(2003), no. 2, 201-215.

[31] P. Kahlig and J. Matkowski, Functional equations involving the logarithmic mean, Z. Angew. Math. Mech., **76**(1996), no. 7, 385-390.

[32] A. O. Pittenger, The logarithmic mean in n variables, Amer. Math. Monthly, **92**(1985), no. 2, 99-104.

[33] G. Polya and G. Szegö, Isoperimetric Inequalities in Mathematical physics, Princeton University Press, Princeton, 1951.

[34] E. B. Leach and M. C. Sholander, Extended mean values II, J. Math. Anal. Appl., **92**(1983), no. 1, 207-223.

[35] J. Sándor, A note on some inequalities for means, Arch. Math., **56**(1991), no. 5, 471-473.

[36] H. Alzer, Ungleichungen für $(e/a)^a(b/e)^b$, Elem. Math., **40**(1985), 120-123.

[37] A. O. Pittenger, Inequalities between arithmetic and logarithmic means, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., (1980), no. 678-715, 15-18.

[38] A. O. Pittenger, The symmetric, logarithmic and power means, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., (1980), no. 678-715, 19-23.

[39] K. B. Stolarsky, The power and generalized logarithmic means, Amer. Math. Monthly, **87**(1980), no. 7, 545-548.

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