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Bounds Improvement for Neuman-Sándor Mean Using Arithmetic, Quadratic and Contraharmonic Means[1]

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Abstract

We prove that the inequalities $\alpha[1/3Q(a,b)+2/3A(a,b)]+(1-\alpha)Q^{1/3}(a,b)A^{2/3}(a,b) < M(a,b) < \beta[1/3Q(a,b)+2/3A(a,b)]+(1-\beta)Q^{1/3}(a,b)A^{2/3}(a,b)$ and $(1-\lambda)C^{1/6}(a,b)A^{5/6}(a,b)+\lambda[1/6C(a,b)+5/6A(a,b)] < M(a,b) < (1-\mu)C^{1/6}(a,b)A^{5/6}(a,b)+\mu[1/6C(a,b)+5/6A(a,b)]$ hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leq(3-3\sqrt[6]{2}\log(1+\sqrt{2}))/[(2+\sqrt{2}-3\sqrt[6]{2})\log(1+\sqrt{2})]=0.777\cdots$, $\beta\geq 4/5$, $\lambda\leq(6-6\sqrt[6]{2}\log(1+\sqrt{2}))/(7-6\sqrt[6]{2}\log(1+\sqrt{2}))=0.274\cdots$, and $\mu\geq 8/25$. Here, $M(a,b)$, $A(a,b)$, $C(a,b)$, and $Q(a,b)$ denote the Neuman-Sándor, arithmetic, contraharmonic, and quadratic means of a and b, respectively.

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1. Introduction

For $a, b > 0$ with $a \neq b$ the Neuman-Sándor mean $M(a, b)$ [1] is defined by

$$M(a,b) = \frac{a-b}{2\text{arcsinh}\left[(a-b)/(a+b)\right]}. \tag{1.1}$$

Recently, the Neuman-Sándor mean has been the subject intensive research. In particular, many remarkable inequalities for the Neuman-Sándor mean $M(a, b)$ can be found in the literature [1-10].

Let $H(a,b) = 2ab/(a+b)$, $G(a,b) = \sqrt{ab}$, $L(a,b) = (b-a)/(\log b - \log a)$, $P(a,b) = (a-b)/(4\arctan\sqrt{a/b} - \pi)$, $I(a,b) = 1/e(b^b/a^a)^{1/(b-a)}$, $A(a,b) = (a+b)/2$, $T(a,b) = (a-b)/\left[2\arcsin((a-b)/(a+b))\right]$, $Q(a,b) = \sqrt{(a^2+b^2)/2}$ and $C(a,b) = (a^2+b^2)/(a+b)$ be the harmonic, geometric, logarithmic, first Seiffert, identric, arithmetic, second Seiffert, quadratic and contraharmonic means of two distinct positive numbers a and b, respectively. Then it is well known that the inequalities

$$H(a,b) < G(a,b) < L(a,b) < P(a,b) < I(a,b)$$

$$< A(a,b) < M(a,b) < T(a,b) < Q(a,b) < C(a,b)$$

hold for all $a, b > 0$ with $a \neq b$.

Neuman and Sándor [1, 2] established that

$$A(a,b) < M(a,b) < \frac{A(a,b)}{\log(1+\sqrt{2})}, \quad \frac{\pi}{4}T(a,b) < M(a,b) < T(a,b),$$

$$\sqrt{A(a,b)T(a,b)} < M(a,b) < \sqrt{\frac{A^2(a,b)T^2(a,b)}{2}},$$

$$M(a,b) < \frac{2A(a,b)+Q(a,b)}{3}, \quad M(a,b) < \frac{A^2(a,b)}{P(a,b)}$$

for all $a, b > 0$ with $a \neq b$.

Let $0 < a, b < 1/2$ with $a \neq b$, $a' = 1-a$ and $b' = 1-b$. Then the following Ky Fan inequalities

$$\frac{G(a,b)}{G(a',b')} < \frac{L(a,b)}{L(a',b')} < \frac{P(a,b)}{P(a',b')} < \frac{A(a,b)}{A(a',b')} < \frac{M(a,b)}{M(a',b')} < \frac{T(a,b)}{T(a',b')}$$

were presented in [1].

Li et al. [3] proved that $L_{p_0}(a,b) < M(a,b) < L_2(a,b)$ for all $a, b > 0$ with $a \neq b$, where $L_p(a,b) = [(b^{p+1} - a^{p+1})/((p+1)(b-a))]^{1/p} (p \neq -1, 0)$, $L_0(a,b) = I(a,b)$ and $L_{-1}(a,b) = L(a,b)$ is the pth generalized logarithmic mean of a and b, and $p_0 = 1.843\cdots$ is the unique solution of the equation $(p+1)^{1/p} = 2\log(1+\sqrt{2})$.

In [4], Neuman established that

$$Q^{1/3}(a,b)A^{2/3}(a,b) < M(a,b) < \frac{1}{3}Q(a,b) + \frac{2}{3}A(a,b) \tag{1.2}$$

and

$$C^{1/6}(a,b)A^{5/6}(a,b) < M(a,b) < \frac{1}{6}C(a,b) + \frac{5}{6}A(a,b) \tag{1.3}$$

for all $a, b > 0$ with $a \neq b$.

In [5, 6], the authors presented the best possible constants α_1, β_1, α_2, β_2, α_3, β_3, α_4, β_4, α_5 and β_5 such that the double inequalities

$$\alpha_1 H(a,b) + (1-\alpha_1)Q(a,b) < M(a,b) < \beta_1 H(a,b) + (1-\beta_1)Q(a,b),$$

$$\alpha_2 G(a,b) + (1-\alpha_2)Q(a,b) < M(a,b) < \beta_2 G(a,b) + (1-\beta_2)Q(a,b),$$

$$\alpha_3 H(a,b) + (1-\alpha_3)C(a,b) < M(a,b) < \beta_3 H(a,b) + (1-\beta_3)C(a,b),$$

$$M_{\alpha_4}(a,b) < M(a,b) < M_{\beta_4}(a,b),$$

$$\alpha_5 I(a,b) < M(a,b) < \beta_5 I(a,b)$$

hold for all $a, b > 0$ with $a \neq b$.

The aim of this paper is to improve and refine inequalities (1.2) and (1.3). Our main results are the following Theorems 1.1 and 1.2.

Theorem 1.1. The double inequality

$$\begin{aligned} \alpha[1/3Q(a,b) + 2/3A(a,b)] + (1-\alpha)Q^{1/3}(a,b)A^{2/3}(a,b) &< M(a,b) \\ < \beta[1/3Q(a,b) + 2/3A(a,b)] + (1-\beta)Q^{1/3}(a,b)A^{2/3}(a,b)& \end{aligned} \tag{1.4}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq (3 - 3\sqrt[6]{2}\log(1+\sqrt{2}))/[(2 + \sqrt{2} - 3\sqrt[6]{2})\log(1+\sqrt{2})] = 0.777\cdots$ and $\beta \geq 4/5$.

Theorem 1.2. The double inequality

$$\begin{aligned}\lambda[1/6C(a,b)+5/6A(a,b)]+(1-\lambda)C^{1/6}(a,b)A^{5/6}(a,b)<M(a,b)\\ <\mu[1/6C(a,b)+5/6A(a,b)]+(1-\mu)C^{1/6}(a,b)A^{5/6}(a,b)\end{aligned} \quad (1.5)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\lambda\leq(6-6\sqrt[6]{2}\log(1+\sqrt{2}))/(7-6\sqrt[6]{2}\log(1+\sqrt{2}))=0.274\cdots$ and $\mu\geq 8/25$.

2. Lemmas

In order to establish our main results we need two Lemmas, which we present in this section.

Lemma 2.1. Let $p\in(0,1)$,

$$f_p(t)=\operatorname{arcsinh}(\sqrt{t^6-1})-\frac{3\sqrt{t^6-1}}{pt^3+3(1-p)t+2p}, \quad (2.1)$$

and $\alpha_0=(3-3\sqrt[6]{2}\log(1+\sqrt{2}))/[(2+\sqrt{2}-3\sqrt[6]{2})\log(1+\sqrt{2})]=0.777\cdots$. Then $f_{4/5}(t)>0$ and $f_{\alpha_0}(t)<0$ for all $t\in(1,\sqrt[6]{2})$.

Proof. From (2.1) one has

$$f_p(1)=0, \quad (2.2)$$

$$f_p(\sqrt[6]{2})=\ln(1+\sqrt{2})-\frac{3}{p(\sqrt{2}+2)+3\sqrt[6]{2}(1-p)}, \quad (2.3)$$

$$f_p{}'(t)=\frac{3(t-1)^2g_p(t)}{[pt^3+3(1-p)t+2p]^2\sqrt{t^6-1}}, \quad (2.4)$$

where

$$\begin{aligned}g_p(t)=&p^2t^6+2p^2t^5+3(-p^2+4p-2)t^4+2(-2p^2+9p-6)t^3\\ &+(4p^2+6p-9)t^2-6(1-p)t-3(1-p).\end{aligned} \quad (2.5)$$

We dived the proof into two cases.

Case 1 $p=4/5$. Then (2.5) becomes

$$g_p(t)=g_{4/5}(t)=\frac{t-1}{25}\left(16t^5+48t^4+90t^3+86t^2+45t+15\right)>0 \quad (2.6)$$

for $t\in(1,\sqrt[6]{2})$.

Therefore, $f_{4/5}(t)>0$ for all $t\in(1,2^{1/6})$ follows from (2.2), (2.4) and (2.6).

Case 2 $p = \alpha_0$. Then (2.3) and (2.5) lead to

$$f_{\alpha_0}(\sqrt[6]{2}) = 0 \tag{2.7}$$

and

$$\begin{aligned} g_p(t) = g_{\alpha_0}(t) =& \alpha_0^2 t^6 + 2\alpha_0^2 t^5 + 3(-\alpha_0^2 + 4\alpha_0 - 2)t^4 - 2(2\alpha_0^2 - 9\alpha_0 + 6)t^3 \\ & - (-4\alpha_0^2 - 6\alpha_0 + 9)t^2 - 6(1-\alpha_0)t - 3(1-\alpha_0). \end{aligned}$$

Note that

$$g_{\alpha_0}(1) = 9(5\alpha_0 - 4) < 0, \quad g_{\alpha_0}(\sqrt[6]{2}) = 0.569\cdots > 0, \tag{2.8}$$

$$\begin{aligned} {g_{\alpha_0}}'(t) =& 6\alpha_0^2 t^5 + 10\alpha_0^2 t^4 + 12(-\alpha_0^2 + 4\alpha_0 - 2)t^3 - 6(2\alpha_0^2 - 9\alpha_0 + 6)t^2 \\ & - 2(-4\alpha_0^2 - 6\alpha_0 + 9)t - 6(1-\alpha_0) \\ >& 6\alpha_0^2 t^2 + 10\alpha_0^2 t^2 + 12(-\alpha_0^2 + 4\alpha_0 - 2)t^2 - 6(2\alpha_0^2 - 9\alpha_0 + 6)t^2 \\ & - 2(-4\alpha_0^2 - 6\alpha_0 + 9)t^2 - 6(1-\alpha_0) \\ =& 6(19\alpha_0 - 13)t^2 - 6(1-\alpha_0) > 12(10\alpha_0 - 7) > 0 \end{aligned} \tag{2.9}$$

for $t \in (1, \sqrt[6]{2})$.

The inequality (2.9) implies that $g_{\alpha_0}(t)$ is strictly increasing in $(1, \sqrt[6]{2})$. Then from (2.4) and (2.8) we clearly see that there exists $t_0 \in (1, \sqrt[6]{2})$ such that $f_{\alpha_0}(t)$ is strictly decreasing in $[1, t_0]$ and strictly increasing in $[t_0, \sqrt[6]{2}]$.

Therefore, $f_{\alpha_0}(t) < 0$ for $t \in (1, \sqrt[6]{2})$ follows from (2.2) and (2.7) together with the piecewise monotonicity of $f_{\alpha_0}(t)$. □

Lemma 2.2. Let $p \in (0, 1)$,

$$F_p(t) = \operatorname{arcsinh}(\sqrt{t^6 - 1}) - \frac{6\sqrt{t^6 - 1}}{pt^6 + 6(1-p)t + 5p}, \tag{2.10}$$

and $\lambda_0 = 6(1 - \sqrt[6]{2}\log(1+\sqrt{2}))/(7 - 6\sqrt[6]{2}\log(1+\sqrt{2})) = 0.274\cdots$. Then $F_{8/25}(t) > 0$ and $F_{\lambda_0}(t) < 0$ for all $t \in (1, \sqrt[6]{2})$.

Proof. From (2.10) we have

$$F_p(1) = 0, \tag{2.11}$$

$$F_p(\sqrt[6]{2}) = \ln(1+\sqrt{2}) - \frac{6}{7p + 6\sqrt[6]{2}(1-p)}, \tag{2.12}$$

$$F_p{}'(t) = \frac{3(t-1)^2 G_p(t)}{[pt^6 + 6(1-p)t + 5p]^2\sqrt{t^6 - 1}}, \tag{2.13}$$

where

$$\begin{aligned} G_p(t) =& p^2t^{12} + 2p^2t^{11} + 3p^2t^{10} + 2p(3+2p)t^9 + p(12+5p)t^8 + 6p(5-p)t^7 \\ &+ p(48-7p)t^6 + 2p(33-4p)t^5 + 3(-3p^2+36p-8)t^4 \\ &+ 2(-5p^2+54p-24)t^3 + (25p^2+36p-36)t^2 \\ &- 24(1-p)t - 12(1-p). \end{aligned} \tag{2.14}$$

We dived the proof into two cases.

Case 1 $p = 8/25$. Then (2.14) becomes

$$\begin{aligned} G_p(t) = G_{8/25}(t) = \frac{4(t-1)}{625}(16t^{11} + 48t^{10} + 96t^9 + 460t^8 + 1140t^7 \\ + 2544t^6 + 4832t^5 + 8004t^4 + 9510t^3 + 7250t^2 + 3825t + 1275) > 0 \end{aligned} \tag{2.15}$$

for $t \in (1, \sqrt[6]{2})$.

Therefore, $F_{8/25}(t) > 0$ for all $t \in (1, \sqrt[6]{2})$ follows from (2.11), (2.13) and (2.15).

Case 2 $p = \lambda_0$. Then (2.12) and (2.14) lead to

$$F_{\lambda_0}(\sqrt[6]{2}) = 0 \tag{2.16}$$

and

$$\begin{aligned} G_p(t) = G_{\lambda_0}(t) =& \lambda_0^2t^{12} + 2\lambda_0^2t^{11} + 3\lambda_0^2t^{10} + 2\lambda_0(3+2\lambda_0)t^9 + \lambda_0(12+5\lambda_0)t^8 \\ &+ 6\lambda_0(5-\lambda_0)t^7 + \lambda_0(48-7\lambda_0)t^6 + 2\lambda_0(33-4\lambda_0)t^5 \\ &+ 3(-3\lambda_0^2+36\lambda_0-8)t^4 + 2(-5\lambda_0^2+54\lambda_0-24)t^3 \\ &+ (25\lambda_0^2+36\lambda_0-36)t^2 - 24(1-\lambda_0)t - 12(1-\lambda_0). \end{aligned}$$

Note that

$$G_{\lambda_0}(1) = 18(25\lambda_0 - 8) < 0, \quad G_{\lambda_0}(\sqrt[6]{2}) = 12.313\cdots > 0, \tag{2.17}$$

$$\begin{aligned} {G_{\lambda_0}}'(t) =& 12\lambda_0^2t^{11} + 22\lambda_0^2t^{10} + 30\lambda_0^2t^9 + 18\lambda_0(3+2\lambda_0)t^8 + 8\lambda_0(12+5\lambda_0)t^7 \\ &+ 42\lambda_0(5-\lambda_0)t^6 + 6\lambda_0(48-7\lambda_0)t^5 + 10\lambda_0(33-4\lambda_0)t^4 \\ &+ 12(-3\lambda_0^2+36\lambda_0-8)t^3 - 6(5\lambda_0^2-54\lambda_0+24)t^2 \\ &- 2(-25\lambda_0^2-36\lambda_0+36)t - 24(1-\lambda_0) \\ >& 12\lambda_0^2t^2 + 22\lambda_0^2t^2 + 30\lambda_0^2t^2 + 18\lambda_0(3+2\lambda_0)t^2 + 8\lambda_0(12+5\lambda_0)t^2 \\ &+ 42\lambda_0(5-\lambda_0)t^2 + 6\lambda_0(48-7\lambda_0)t^2 + 10\lambda_0(33-4\lambda_0)t^2 \\ &+ 12(-3\lambda_0^2+36\lambda_0-8)t^2 - 6(5\lambda_0^2-54\lambda_0+24)t^2 \\ &- 2(-25\lambda_0^2-36\lambda_0+36)t^2 - 24(1-\lambda_0) \\ =& (1806\lambda_0 - 312)t^2 - 24(1-\lambda_0) > 1830\lambda_0 - 336 > 0. \end{aligned} \tag{2.18}$$

for $t \in (1, \sqrt[6]{2})$.

The inequality (2.18) implies that $G_{\lambda_0}(t)$ is strictly increasing in $[1, \sqrt[6]{2}]$. Then from (2.13) and (2.17) we clearly see that there exists $t_1 \in (1, \sqrt[6]{2})$ such that $F_{\lambda_0}(t)$ is strictly decreasing in $[1, t_1]$ and strictly increasing in $[t_1, \sqrt[6]{2}]$.

Therefore, $F_{\lambda_0}(t) < 0$ for all $t \in (1, \sqrt[6]{2})$ follows from (2.11) and (2.16) together with the piecewise monotonicity of $F_{\lambda_0}(t)$. □

3. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. From (1.1) we clearly see that $M(a, b)$, $Q(a, b)$ and $A(a, b)$ are symmetric and homogeneous of degree 1. Without loss of generality, we assume that $a > b$. Let $p \in (0, 1)$ and $x = (a-b)/(a+b)$, $t = \sqrt[6]{x^2+1}$ and $\alpha_0 = (3 - 3\sqrt[6]{2}\log(1+\sqrt{2}))/[(2+\sqrt{2}-3\sqrt[6]{2})\log(1+\sqrt{2})] = 0.777\cdots$. Then $x \in (0, 1)$, $t \in (1, \sqrt[6]{2})$,

$$\begin{aligned}&\frac{M(a,b) - Q^{1/3}(a,b)A^{2/3}(a,b)}{1/3Q(a,b) + 2A(a,b)/3 - Q^{1/3}(a,b)A^{2/3}(a,b)}\\ =&\frac{3[x - \sqrt[6]{1+x^2}\text{arcsinh}(x)]}{(\sqrt{1+x^2} - 3\sqrt[6]{1+x^2} + 2)\text{arcsinh}(x)}\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}&p\left[\frac{1}{3}Q(a,b) + \frac{2}{3}A(a,b)\right] + (1-p)Q^{1/3}(a,b)A^{2/3}(a,b) - M(a,b)\\ =&A(a,b)\left[p(\frac{1}{3}\sqrt{1+x^2} + \frac{2}{3}) + (1-p)\sqrt[6]{1+x^2} - \frac{x}{\text{arcsinh}(x)}\right]\\ =&\frac{A(a,b)\left[p(\sqrt{1+x^2}+2) + 3(1-p)\sqrt[6]{1+x^2}\right]}{3\text{arcsinh}(x)} f_p(t).\end{aligned} \tag{3.2}$$

where $f_p(t)$ is defined as in Lemma 2.1. Note that

$$\lim_{x\to 0}\frac{3[x - \sqrt[6]{1+x^2}\text{arcsinh}(x)]}{(\sqrt{1+x^2} - 3\sqrt[6]{1+x^2} + 2)\text{arcsinh}(x)} = \frac{4}{5}, \tag{3.3}$$

$$\lim_{x\to 1}\frac{3[x - \sqrt[6]{1+x^2}\text{arcsinh}(x)]}{(\sqrt{1+x^2} - 3\sqrt[6]{1+x^2} + 2)\text{arcsinh}(x)} = \alpha_0. \tag{3.4}$$

Therefore, Theorem 1.1 follows easily from (3.2)-(3.4) and Lemma 2.1. □

Proof of Theorem 1.2. Since $M(a, b)$, $C(a, b)$ and $A(a, b)$ are symmetric and homogeneous of degree 1. Without loss of generality, we assume that $a > b$. Let $p \in (0, 1)$ and $x = (a-b)/(a+b)$, $t = \sqrt[6]{x^2+1}$ and $\lambda_0 =$

$6(1-\sqrt[6]{2}\log(1+\sqrt{2}))/(7-6\sqrt[6]{2}\log(1+\sqrt{2})) = 0.274\cdots$. Then $x\in(0,1)$, $t\in(1,\sqrt[6]{2})$,

$$\begin{aligned}&\frac{M(a,b)-C^{1/6}(a,b)A^{5/6}(a,b)}{1/6C(a,b)+5A(a,b)/6-C^{1/6}(a,b)A^{5/6}(a,b)}\\ =&\frac{6[x-\sqrt[6]{1+x^2}\mathrm{arcsinh}(x)]}{(x^2+6-6\sqrt[6]{1+x^2})\mathrm{arcsinh}(x)}\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&p\left[\frac{1}{6}C(a,b)+\frac{5}{6}A(a,b)\right]+(1-p)C^{1/6}(a,b)A^{5/6}(a,b)-M(a,b)\\ =&A(a,b)\left[p\left(1+\frac{1}{6}x^2\right)+(1-p)\sqrt[6]{1+x^2}-\frac{x}{\mathrm{arcsinh}(x)}\right]\\ =&\frac{A(a,b)\left[p(6+x^2)+6(1-p)\sqrt[6]{1+x^2}\right]}{6\mathrm{arcsinh}(x)}F_p(t).\end{aligned} \tag{3.6}$$

where $F_p(t)$ is defined as in Lemma 2.2. Note that

$$\lim_{x\to 0}\frac{6[x-\sqrt[6]{1+x^2}\mathrm{arcsinh}(x)]}{(x^2+6-6\sqrt[6]{1+x^2})\mathrm{arcsinh}(x)}=\frac{8}{25}, \tag{3.7}$$

$$\lim_{x\to 1}\frac{6[x-\sqrt[6]{1+x^2}\mathrm{arcsinh}(x)]}{(x^2+6-6\sqrt[6]{1+x^2})\mathrm{arcsinh}(x)}=\lambda_0. \tag{3.8}$$

Therefore, Theorem 1.2 follows easily from (3.5)-(3.8) and Lemma 2.2. □

Remark 3.1. If we take $\alpha=0$ and $\beta=1$ in Theorem 1.1, then the double inequality (1.4) reduces to (1.2).

Remark 3.2. If we take $\lambda=0$ and $\mu=1$ in Theorem 1.2, then the double inequality (1.5) reduces to (1.3).

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