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Bounds for the Neuman-Sándor Mean in Terms of Logarithmic, Quadratic or Contraharmonic Means[1]

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Abstract

We present the best possible parameters α_1, α_2, β_1, β_2, such that the double inequalities $L^{\alpha_1}(a,b)Q^{1-\alpha_1}(a,b) < M(a,b) < L^{\beta_1}(a,b)Q^{1-\beta_1}(a,b)$ and $L^{\alpha_2}(a,b)C^{1-\alpha_2}(a,b) < M(a,b) < L^{\beta_2}(a,b)\ C^{1-\beta_2}(a,b)$ hold for all $a,b > 0$, where $L(a,b)$, $M(a,b)$, $Q(a,b)$ and $C(a,b)$ are the logarithmic, Neuman-Sándor, quadratic and contraharmonic means of a and b, respectively.

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1. Introduction

For $a, b > 0$ with $a \neq b$ the Neuman-Sándor mean $M(a,b)$ [1] is defined by

$$M(a,b) = \frac{a-b}{2\sinh^{-1}(\frac{a-b}{a+b})}$$

where $\sinh^{-1}(x) = \log(x + \sqrt{x^2+1})$ is the inverse hyperbolic sine function.

Recently, the Neuman-Sándor mean has been the subject intensive research. In particular, many remarkable inequalities for the Neuman-Sándor mean can be found in the literature [1, 2, 3, 4, 5, 6].

Let $H(a,b) = 2ab/(a+b)$, $G(a,b) = \sqrt{ab}$, $L(a,b) = (b-a)/(\log b - \log a)$, $P(a,b) = (a-b)/(4\arctan\sqrt{a/b} - \pi)$, $I(a,b) = 1/e(b^b/a^a)^{1/(b-a)}$, $A(a,b) = (a+b)/2$, $T(a,b) = (a-b)/[2\arctan((a-b)/(a+b))]$, $Q(a,b) = \sqrt{(a^2+b^2)/2}$ and $C(a,b) = (a^2+b^2)/(a+b)$ be the harmonic, geometric, logarithmic, first Seiffert, identric, arithmetic, second Seiffert, quadratic and contraharmonic means of two distinct positive numbers a and b, respectively. Then it is well known that the inequalities

$$\begin{aligned} &H(a,b) < G(a,b) < L(a,b) < P(a,b) < I(a,b) \\ &< A(a,b) < M(a,b) < T(a,b) < Q(a,b) < C(a,b), \end{aligned} \tag{1.1}$$

hold for all $a, b > 0$ with $a \neq b$.

Neuman and Sándor [1, 2] established that

$$A(a,b) < M(a,b) < \frac{A(a,b)}{\log(1+\sqrt{2})}, \quad \frac{\pi}{4}T(a,b) < M(a,b) < T(a,b),$$
$$\sqrt{A(a,b)T(a,b)} < M(a,b) < \sqrt{\frac{A^2(a,b)+T^2(a,b)}{2}},$$
$$M(a,b) < \frac{A^2(a,b)}{P(a,b)}, \quad M(a,b) < \frac{2A(a,b)+Q(a,b)}{3}$$

for all $a, b > 0$ with $a \neq b$.

Let $0 < a, b < 1/2$ with $a \neq b, a' = 1-a$ and $b' = 1-b$. Then the ky Fan inequalities

$$\frac{G(a,b)}{G(a',b')} < \frac{L(a,b)}{L(a',b')} < \frac{P(a,b)}{P(a',b')} < \frac{A(a,b)}{A(a',b')} < \frac{M(a,b)}{M(a',b')} < \frac{T(a,b)}{T(a',b')}$$

were presented in [1].

Li et al. [7] proved that the double inequality $L_{p_0}(a,b) < M(a,b) < L_2(a,b)$ holds for all $a, b > 0$ with $a \neq b$, where $L_p(a,b) = [(b^{p+1} - a^{p+1})/((p+1)(b-a))]^{1/p}$ $(p \neq -1, 0)$, $L_0(a,b) = I(a,b)$ and $L_1(a,b) = L(a,b)$ is the pth generalized logarithmic mean of a and b, and $p_0 = 1.843....$ is the unique solution of the equation $(p+1)^{1/p} = 2\log(1+\sqrt{2})$

In [8], Neuman proved that the double inequalities

$$Q^{\alpha}(a,b)A^{1-\alpha}(a,b) < M(a,b) < Q^{\beta}(a,b)A^{1-\beta}(a,b)$$

and

$$C^{\lambda}(a,b)A^{1-\lambda}(a,b) < M(a,b) < C^{\mu}(a,b)A^{1-\mu}(a,b)$$

hold for all $a, b > 0$ with $a \neq b$ if $\alpha \leq 1/3$, $\beta \geq 2[\log(2+\sqrt{2}) - \log 3]/\log 2$, $\lambda \leq 1/6$ and $\mu \geq [\log(2+\sqrt{2}) - \log 3]/\log 2$.

Zhao et al. [9] found the least values $\alpha_1, \alpha_2, \alpha_3$ and the greatest values $\beta_1, \beta_2, \beta_3$, such that the double inequalities

$$\alpha_1 H(a,b) + (1-\alpha_1)Q(a,b) < M(a,b) < \beta_1 H(a,b) + (1-\beta_1)Q(a,b),$$
$$\alpha_2 G(a,b) + (1-\alpha_2)Q(a,b) < M(a,b) < \beta_2 G(a,b) + (1-\beta_2)Q(a,b),$$
$$\alpha_3 H(a,b) + (1-\alpha_3)C(a,b) < M(a,b) < \beta_3 H(a,b) + (1-\beta_3)C(a,b)$$

hold for all $a, b > 0$ with $a \neq b$.

In [10] the authors answer the question: What are the best possible constants α, β, λ and μ, such that the double inequalities $M_{\alpha}(a,b) < M(a,b) < M_{\beta}(a,b)$ and $\lambda I(a,b) < M(a,b) < \mu I(a,b)$ hold for all $a, b > 0$ with $a \neq b$, where $M_p(a,b) = ((a^p + b^p)/2)^{1/p} (p \neq 0)$ and $M_0(a,b) = \sqrt{ab}$ is the pth power mean of a and b.

The aim of this paper is to find the best possible parameters $\alpha_1, \alpha_2, \beta_1$ and β_2 such that the double inequalities $L^{\alpha_1}(a,b)Q^{1-\alpha_1}(a,b) < M(a,b) < L^{\beta_1}(a,b)Q^{1-\beta_1}(a,b)$ and $L^{\alpha_2}(a,b)C^{1-\alpha_2}(a,b) < M(a,b) < L^{\beta_2}(a,b)\ C^{1-\beta_2}(a,b)$ hold for all $a, b > 0$ with $a \neq b$. All numerical computations are carried out using MATHEMATICA software.

2. Lemmas

In order to establish our main results we need several lemmas, which we present in this section.

The following Lemma 2.1 follows immediately from the power series expansions for the inverse hyperbolic tangent, inverse hyperbolic sine functions, and the formula for the sum of the infinite geometric series.

Lemma 2.1. The inequalities

$$\log\left(\frac{1+x}{1-x}\right) > 2x + \frac{2x^3}{3}, \tag{2.1}$$

$$x - \frac{x^3}{6} < \sinh^{-1}(x) < x, \tag{2.2}$$

$$1 - x^2 + x^4 - x^6 < \frac{1}{1+x^2} < 1 - x^2 + x^4 \tag{2.3}$$

hold true for $x \in (0,1)$. □

Lemma 2.2. The inequality

$$\frac{1}{\sqrt{1+x^2}\sinh^{-1}(x)} < \frac{1}{x} - \frac{x}{3} + \frac{11x^3}{45} - \frac{x^5}{15}$$

holds for $x \in (0,1)$.

Proof. Let

$$f(x) = x - \left(1 - \frac{x^2}{3} + \frac{11x^4}{45} - \frac{x^6}{15}\right)\sqrt{1+x^2}\sinh^{-1}(x).$$

Then

$$\frac{1}{\sqrt{1+x^2}\sinh^{-1}(x)} - \left(\frac{1}{x} - \frac{x}{3} + \frac{11x^3}{45} - \frac{x^5}{15}\right) = \frac{f(x)}{x\sqrt{1+x^2}\sinh^{-1}(x)}, \tag{2.4}$$

$$f(0) = 0, \tag{2.5}$$

$$f'(x) = \frac{xf_1(x)}{45\sqrt{1+x^2}}, \tag{2.6}$$

where

$$f_1(x) = x(15 - 11x^2 + 3x^4)\sqrt{1+x^2} - (15 - x^2 + 37x^4 - 21x^6)\sinh^{-1}(x). \tag{2.7}$$

It follows from (2.2) and (2.7) together with $\sqrt{1+x^2} < 1 + x^2/2$ that

$$\begin{aligned} f_1(x) &< x(15 - 11x^2 + 3x^4)\left(1 + \frac{x^2}{2}\right) - (15 - x^2 + 37x^4 - 21x^6)\left(x - \frac{x^3}{6}\right) \\ &= -\frac{x^5}{6}(238 - 172x^2 + 21x^4) < 0 \end{aligned} \tag{2.8}$$

for $x \in (0,1)$.

Therefore, Lemma 2.2 follows from (2.4)-(2.6) and (2.8). □

Lemma 2.3. The inequality

$$\frac{1}{(1-x^2)\log[(1+x)/(1-x)]} > \frac{1}{2x} + \frac{x}{3} + \frac{13x^3}{45}$$

holds for $x \in (0,1)$.

Proof. Let

$$h(x) = x - \left(\frac{1}{2} + \frac{x^2}{3} + \frac{13x^4}{45}\right)(1-x^2)\log\left(\frac{1+x}{1-x}\right).$$

Then

$$\begin{aligned} &\frac{1}{(1-x^2)\log[(1+x)/(1-x)]} - \left(\frac{1}{2x} + \frac{x}{3} + \frac{13x^3}{45}\right) \\ =&\frac{h(x)}{x(1-x^2)\log[(1+x)/(1-x)]}, \end{aligned} \tag{2.9}$$

$$h(0) = 0, \tag{2.10}$$

$$h'(x) = \frac{xh_1(x)}{45}, \tag{2.11}$$

where

$$h_1(x) = (15 + 8x^2 + 78x^4)\log\left(\frac{1+x}{1-x}\right) - 30x - 26x^3. \tag{2.12}$$

It follows from (2.1) and (2.12) that

$$\begin{aligned} h_1(x) &> (15 + 8x^2 + 78x^4)(2x + \frac{2x^3}{3}) - 30x - 26x^3 \\ &= \frac{x^5}{3}(484 + 156x^2) > 0 \end{aligned} \tag{2.13}$$

for $x \in (0,1)$.

Therefore, Lemma 2.3 follows from (2.9)-(2.11) and (2.13). □

3. Main Results

Theorem 3.1. The double inequality

$$L^{\alpha_1}(a,b)Q^{1-\alpha_1}(a,b) < M(a,b) < L^{\beta_1}(a,b)Q^{1-\beta_1}(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \geq 2/5$ and $\beta_1 \leq 0$.

Proof. Since $L(a,b)$, $M(a,b)$ and $Q(a,b)$ are symmetric and homogeneous of degree one. Without loss generality, we assume that $a > b$. Let $x = (a-b)/(a+b)$, then $x \in (0,1)$,

$$\frac{L(a,b)}{A(a,b)} = \frac{2x}{\log[(1+x)/(1-x)]}, \frac{M(a,b)}{A(a,b)} = \frac{x}{\sinh^{-1}(x)}, \frac{Q(a,b)}{A(a,b)} = \sqrt{1+x^2}, \tag{3.1}$$

$$\frac{\log Q(a,b) - \log M(a,b)}{\log Q(a,b) - \log L(a,b)} = \frac{\log(1+x^2) - 2\log[x/\sinh^{-1}(x)]}{\log(1+x^2) - 2\log(2x/\log[(1+x)/(1-x)])}, \tag{3.2}$$

$$\lim_{x\to 0^+} \frac{\log(1+x^2) - 2\log[x/\sinh^{-1}(x)]}{\log(1+x^2) - 2\log(2x/\log[(1+x)/(1-x)])} = \frac{2}{5}, \tag{3.3}$$

$$\lim_{x\to 1^-} \frac{\log(1+x^2) - 2\log[x/\sinh^{-1}(x)]}{\log(1+x^2) - 2\log(2x/\log[(1+x)/(1-x)])} = 0, \tag{3.4}$$

$$\begin{aligned} &\frac{2}{5}\log L(a,b) + \frac{3}{5}\log Q(a,b) - \log M(a,b) \\ =&\frac{3}{10}\log(1+x^2) + \log(\sinh^{-1}(x)) - \frac{2}{5}\log\left(\log\frac{1+x}{1-x}\right) - \frac{3}{5}\log x + \frac{2}{5}\log 2 \\ :=&F(x). \end{aligned} \tag{3.5}$$

It follows from Lemmas 2.2 and 2.3 together with (2.3) and (3.5) that

$$F(0^+) = 0, \tag{3.6}$$

$$\begin{aligned} F'(x) &= \frac{1}{\sqrt{1+x^2}\sinh^{-1}(x)} - \frac{4}{5(1-x^2)\log[(1+x)/(1-x)]} - \frac{3}{5x(1+x^2)} \\ &< \frac{1}{x} - \frac{x}{3} + \frac{11x^3}{45} - \frac{x^5}{15} - \frac{4}{5}\left(\frac{1}{2x} + \frac{x}{3} + \frac{13x^3}{45}\right) - \frac{3}{5x}(1 - x^2 + x^4 - x^6) \\ &= -\frac{8x^3}{15}\left(\frac{11}{10} - x^2\right) < 0 \end{aligned} \tag{3.7}$$

for $x \in (0,1)$.

From (3.5)-(3.7) one has

$$M(a,b) > L^{2/5}(a,b)Q^{3/5}(a,b). \tag{3.8}$$

Inequalities (1.1) and (3.8) in conjunction with the following statements give the asserted result.

• If $\alpha_1 < 2/5$, then (3.2) and (3.3) imply that there exists $\delta_1 \in (0,1)$ such that $M(a,b) < L^{\alpha_1}(a,b)Q^{1-\alpha_1}(a,b)$ for all $a,b>0$ with $(a-b)/(a+b) \in (0,\delta_1)$.

•• If $\beta_1 > 0$, then (3.2) and (3.4) imply that there exists $\delta_2 \in (0,1)$ such that $M(a,b) > L^{\beta_1}(a,b)Q^{1-\beta_1}(a,b)$ for all $a,b>0$ with $(a-b)/(a+b) \in (1-\delta_2, 1)$. □

Theorem 3.2. The double inequality

$$L^{\alpha_2}(a,b)C^{1-\alpha_2}(a,b) < M(a,b) < L^{\beta_2}(a,b)C^{1-\beta_2}(a,b)$$

holds for all $a,b>0$ with $a \neq b$ if and only if $\alpha_2 \geq 5/8$ and $\beta_2 \leq 0$.

Proof. We will follow, to some extent, lines in the proof of Theorem 3.1. Since $L(a,b)$, $M(a,b)$ and $C(a,b)$ are symmetric and homogenous of degree one. Without loss generality, we assume that $a>b$. Let $x=(a-b)/(a+b)$, then $x \in (0,1)$. Making use of (3.1) and $C(a,b)/A(a,b) = 1+x^2$ we have

$$\frac{\log C(a,b) - \log M(a,b)}{\log C(a,b) - \log L(a,b)} = \frac{\log(1+x^2) - \log[x/\sinh^{-1}(x)]}{\log(1+x^2) - \log(2x/\log[(1+x)/(1-x)])}, \tag{3.9}$$

$$\lim_{x\to 0^+} \frac{\log(1+x^2) - \log[x/\sinh^{-1}(x)]}{\log(1+x^2) - \log(2x/\log[(1+x)/(1-x)])} = \frac{5}{8}, \tag{3.10}$$

$$\lim_{x\to 1^-} \frac{\log(1+x^2) - \log[x/\sinh^{-1}(x)]}{\log(1+x^2) - \log(2x/\log[(1+x)/(1-x)])} = 0, \tag{3.11}$$

$$\begin{aligned} &\frac{5}{8}\log L(a,b) + \frac{3}{8}\log C(a,b) - \log M(a,b) \\ =&\frac{3}{8}\log(1+x^2) + \log(\sinh^{-1}(x)) - \frac{5}{8}\log\left(\log\frac{1+x}{1-x}\right) - \frac{3}{8}\log x + \frac{2}{8}\log 2 \\ :=&G(x). \end{aligned} \tag{3.12}$$

It follows from Lemma 2.1 and 2.2 together with (2.3) and (3.12) that

$$G(0^+) = 0, \tag{3.13}$$

$$\begin{aligned} G'(x) &= \frac{1}{\sqrt{1+x^2}\sinh^{-1}(x)} - \frac{5}{4(1-x^2)\log[(1+x)/(1-x)]} - \frac{3}{8x} + \frac{3x}{4(1+x^2)} \\ &< \frac{1}{x} - \frac{x}{3} + \frac{11x^3}{45} - \frac{x^5}{15} - \frac{5}{4}\left(\frac{1}{2x} + \frac{x}{3} + \frac{13x^3}{45}\right) - \frac{3}{8x} + \frac{3x}{4}(1-x^2+x^4) \\ &= -\frac{41x^3}{60}\left(\frac{52}{41} - x^2\right) < 0 \end{aligned} \tag{3.14}$$

for $x \in (0,1)$.

From (3.12)-(3.14) we get

$$M(a,b) > L^{5/8}(a,b)C^{3/8}(a,b). \tag{3.15}$$

Inequalities (1.1) and (3.15) in conjunction with the following statements give the asserted result.

• If $\alpha_2 < 5/8$, then (3.9) and (3.10) imply that there exists $\delta_3 \in (0,1)$ such that $M(a,b) < L^{\alpha_2}(a,b)C^{1-\alpha_2}(a,b)$ for all $a,b > 0$ with $(a-b)/(a+b) \in (0,\delta_3)$.

•• If $\beta_2 > 0$, then (3.9) and (3.11) imply that there exists $\delta_4 \in (0,1)$ such that $M(a,b) > L^{\beta_2}(a,b)C^{1-\beta_2}(a,b)$ for all $a,b > 0$ with $(a-b)/(a+b) \in (1-\delta_4, 1)$. □

From the first inequalities in Theorems Theorem 3.1 and Theorem 3.2 we get Corollary 3.3 as follows.

Corollary 3.3. The inequalities

$$\log\frac{1}{M(a,b)} < \frac{2}{5}\log\frac{1}{L(a,b)} + \frac{3}{5}\log\frac{1}{Q(a,b)},$$

$$\log\frac{1}{M(a,b)} < \frac{5}{8}\log\frac{1}{L(a,b)} + \frac{3}{8}\log\frac{1}{C(a,b)},$$

hold for all $a,b > 0$ with $a \neq b$. □

References

[1] E. Neuman, J. Sándor, On the Schwab-Borchardt mean, Math. Panno., **14** (2003), no. 2, 253-266.

[2] E. Neuman and J. Sándor, On the Schwab-Borchardt mean II, Math. Pannon., **17** (2006), no. 1, 49-59.

[3] W. M. Qian and Y. M. Chu, On certain inequalities for Neuman-Sándor mean, Abstr. Appl. Anal. (2013), Art. ID 790783, 6 pp.

[4] Z. Y. He, W. M. Qian, Y. L. Jiang, Y. Q. Song and Y. M. Chu, Bounds for the combinations of Neuman-Sándor, arithmetic, and second Seiffert means in terms of contraharmonic mean, Abstr. Appl. Anal. (2013), Art. ID 903982, 5 pp.

[5] T. H. Zhao, Y. M. Chu, Y. L. Jiang and Y. M. Li, Best possible bounds for Neuman-Sándor mean by the identric, quadratic and contraharmonic means, Abstr. Appl. Anal. (2013), Art. ID 348326, 12 pp.

[6] Y. M. Chu, B. Y. Long, W. M. Gong and Y. Q. Song, Sharp bounds for Seiffert and Neuman-Sándor means in terms of generalized logarithmic means, J. Inequal. Appl. (2013), 2013: 10, 13 pp.

[7] Y. M. Li, B. Y. Long and Y. M. Chu, Sharp bounds for the Neuman-Sándor mean in terms of generalized logarithmic mean, J. Math. Inequal., **6** (2012), no. 54, 567-577.

[8] E. Neuman, A note on a certain bivariate mean, J. Math. Inequal., **6** (2012), no. 4, 637-643.

[9] T. H. Zhao, Y. M. Chu and B. Y. Liu, Optimal bounds for Neuman-Sándor mean in terms of the convex combinations of harmonic, geometric, quadratic, and contraharmonic means, Abstr. Appl. Anal. (2012), Art. ID 302635, 9 pp.

[10] Y. M. Chu and B. Y. Long, Bounds of the Neuman-Sándor mean using power and identric means, Abstr. Appl. Anal. (2013), Art. ID 832591, 6 pp.

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