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Right Derivations on Semirings

S. P. Nirmala Devi

Department of Mathematics
Sree Sowdambika College of Engineering
Aruppukottai - 626101, India
spnirmala1980@gmail.com

M. Chandramouleeswaran

Department of Mathematics
Saiva Bhanu Kshatriya College
Aruppukottai- 626101, India
moulee59@gmail.com

Abstract

Motivated by some works on derivations on rings, Chandramouless-waran and Thiruveni discussed the notion of derivations on semirings. In this paper, we discuss the notion of right derivations on semirings and prove some simple properties.

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1 Introduction

Richard Dedekind used the notion of semiring in his study of ideals without giving the semiring a formal definition. However, H.S. Vandiver,[5] gave a formal definition and introduced the notion of semiring in 1934. A natural example of a semiring which is not a ring is the set of natural numbers in $\mathbb{N}$ under usual addition and multiplication of numbers.

Even though, the study of derivations in rings was initiated long back, it got its significance only after Posner [4] who in 1955 established two very striking results. The notion of derivation has also been generalized in various directions. In [3], Jonathan Golan mentioned about the derivation on a semiring. However, nothing more has been said on it. This motived Chandramouleeswaran and Thiruveni to introduce and discuss the notion of derivation on a semiring and its properties[2]. In 1990, Bresar and Vukman [1] firstly introduced the notion of a left- derivation in a ring and proved that a left- derivation of a semiprime ring R must map R into its center. Motivated by this, in this paper, we introduce the notion of right-derivation on a semiring S and prove some simple properties.

2 Preliminaries

In this section we recall some basic definitions on semirings and derivations on it, that are needed for our work.

Definition 2.1 *A semiring $(S, +, \cdot)$ is an algebraic system with a nonempty set S together with two binary operations $+$ and $\cdot$ such that*

1. *$(S, +)$ is a semi group*
2. *$(S, \cdot)$ is a semi group*
3. *For all $a, b, c \in S$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ hold.*

Definition 2.2 *A semiring $(S, +, \cdot)$ is said to be additively commutative if $(S, +)$ is a commutative semi group. A semiring $(S, +, \cdot)$ is said to be multiplicatively commutative if $(S, \cdot)$ is a commutative semi group. It is said to be commutative if both $(S, +)$ and $(S, \cdot)$ are commutative.*

Definition 2.3 *The semiring $(S, +, \cdot)$ is said to be a semiring with zero, if it has an element 0 in S such that $x + 0 = x = 0 + x$ and $x \cdot 0 = 0 = 0 \cdot x \ \ \forall x \in S$.*

Definition 2.4 *A semiring $(S, +, \cdot)$ is said to be a semiring with an identity element 1, if there exists an element $1 \neq 0 \in S$ such that $1 \cdot x = x = x \cdot 1 \ \forall x \in S$.*

Definition 2.5 *Let $(S, +, \cdot)$ be a semiring. An element α of S is called additively left cancellative if for all $\alpha, \beta, \gamma \in S, \alpha + \beta = \alpha + \gamma \ \Rightarrow \beta = \gamma$. If every element of a semiring S is additively left cancellative, it is called an additively left cancellative semiring.*

Analogously, one can define an additively right cancellative semiring.

Definition 2.6 *A semiring $(S, +, \cdot)$ is said to be additively cancellative if it is both additively left and right cancellative.*

Definition 2.7 *Let S be a semiring. A left $S-$semimodule is a commutative monoid $(M, +, 0_M)$ in which scalar mulitiplication $S \times M \rightarrow M$, denoted by $(s, m) \rightarrow sm$, satisfies the following conditions*

1. $(ss')m = s(s'm)$
2. $s(m + m') = sm + sm'$
3. $(s + s')m = sm + s'm$
4. $1_S m = m$
5. $s0_M = 0_M = 0_S m \ \ \forall\ s, s' \in S \ \ \forall\ m, m' \in M.$

If $V(M) = M$ then M is an $S-$module where $V(M)$ is the set of all elements of M having additive inverse.

Definition 2.8 *Let S be a semiring.*

1. *S is said to be prime if $aSb = 0 \Rightarrow a = 0$ or $b = 0$.*
2. *S is said to be semiprime if $aSa = 0 \ \Rightarrow a = 0$.*
3. *S is said to be 2 torsion free if $2a = 0, a \in S \ \Rightarrow a = 0$.*

Definition 2.9 *Let $(S, +, \cdot)$ be a semiring and X be S module. A derivation on S is a map $d : S \rightarrow X$ satisfying the following conditions*

1. $d(x + y) = d(x) + d(y) \ \ \forall\ x, y \in S$
2. $d(xy) = d(x)y + xd(y), \ \ \forall\ x, y \in S.$

Example 2.10 Let S be a semiring. Let $M_2(S) = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \in S. \right\}$

Define $d : M_2(S) \rightarrow M_2(S)$ is given by $d\left[\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$

Then d is a derivation on $M_2(S)$.

3 Right-Derivations

In this section, we introduce the notion of right-(left-)derivations on semirings, to illustrate the concept ,discuss examples and prove some simple properties.

Definition 3.1 *Let S be a semiring And X be Left-SModule. An additive map $d_L : S \to X$ is said to be a left-derivation if $d_L(xy) = xd_L(y)+yd_L(x) \ \ \forall\, x, y \in S$.*

Example 3.2 Let S be commutative semiring with characteristic 4. Let $M_2(S) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in S. \right\}$

The map $d_L : M_2(S) \to M_2(S)$ given by

$$d_L \left[\begin{pmatrix} a & b \\ b & a \end{pmatrix} \right] = \begin{pmatrix} 0 & 2b \\ 0 & 0 \end{pmatrix}$$

is a left-derivation but not a derivation on $M_2(S)$.

Analogously we can define a right derivation on S as follows.

Definition 3.3 *Let S be a semiring and X be a right-Smodule. An additive map $d_R : S \to X$ is said to be a right-derivation if $d_R(xy) = d_R(x)y + d_R(y)x \ \ \forall\, x, y \in S$.*

Example 3.4 Let S be commutative semiring with characteristic 4. Let $M_2(S) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in S. \right\}$

The map $d_R : M_2(S) \to M_2(S)$ given by

$$d_R \left[\begin{pmatrix} a & b \\ b & a \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 \\ 2b & 0 \end{pmatrix}$$

is a right-derivation but not a derivation on $M_2(S)$.

One can easily prove the following.

Lemma 3.5

1. *Sum of two left derivations on an additively commutative semiring is again a left derivation.*

2. *Sum of two right derivations on an additively commutative semiring is again a right derivation.*

Lemma 3.6 *Let S be a semiring, X be a righr-Smodule and $d_R : S \to X$ be a right- derivation. For any element $a \in S, d_R(a^n) = nd_R(a)a^{n-1}$.*

Proof: We prove this result by induction hypothesis to prove this result.

Take $n = 2$. Then
$d_R(a^2) = d_R(a \cdot a) = d_R(a)a + d_R(a)a = 2ad_R(a)a$.
Assume that $d_R(a^n) = nd_R(a)a^{n-1}$. Now,

$$\begin{aligned}
d_R(a^{n+1}) &= d_R(a^n \cdot a) \\
&= d_R(a^n)a + d_R(a)a^n \\
&= [nd_R(a)a^{n-1}]a + d_R(a)a^n \\
&= nd_R(a)a^n + d_R(a)a^n \\
&= (n+1)d_R(a)a^{n+1} \\
d_R(a^n) &= nd_R(a)a^{n-1} \quad \forall\ n > 0.
\end{aligned}$$

Analogously we can prove the following.

Lemma 3.7 *Let S be a semiring, X be a left-Smodule and $d_L : S \to X$ be a Left- derivation. For any element $a \in S, d_L(a^n) = na^{n-1}d_L(a)$.*

Theorem 3.8 *Let S be a semiring and X be a 2-torsion free right-Smodule. If $d_R : S \to X$ is a right-derivation then $a, b, c \in R$, then*

1. $d_R(ab + ba) = 2d_R(b)a + 2d_R(a)b$.
2. $d_R(aba) = d_R(b)a^2 + 3d_R(a)ba - d_R(a)ab$.
3. $d_R(abc+cba) = d_R(a)(cb+bc)+3d_R(b)ca+3d_R(c)ba-d_R(c)ab-d_R(b)ac$.
4. $d_R(a)a(ba - ab) = d_R(a)(ab - ba)a$.
5. $(ba - ab)(d_R(ba) - d_R(a)b - d_R(b)a) = 0$.

Proof:

$$\begin{aligned}
d_R(ab) &= d_R(a)b + d_R(b)a \\
d_R(ab + ba) &= d_R(ab) + d_R(ba) \\
&= d_R(a)b + d_R(b)a + d_R(b)a + d_R(a)b \\
&= 2d_R(a)b + 2d_R(b)a \quad \cdots\cdots (1)
\end{aligned}$$

Replace b by $ab + ba$ in (1),

$$\begin{aligned}
d_R(a(ab + ba) + (ab + ba)a) &= 2d_R(a)(ab + ba) + 2d_R(ab + ba)a \\
&= 2d_R(a)ab + 6d_R(a)ba + 4d_R(b)a^2 \quad \cdots\cdots (2)
\end{aligned}$$

$$\begin{aligned}
d_R(a^2b + aba + aba + ba^2) &= d_R(a^2b) + 2d_R(aba) + d_R(ba^2) \\
&= d_R(a^2)b + d_R(b)a^2 + 2d_R(aba) + d_R(b)a^2 + d_R(a^2)b \\
&= 2d_R(b)a^2 + 4d_R(a)ab + 2d_R(aba) \cdots\cdots (3)
\end{aligned}$$

From (2) and (3)

$$\begin{aligned} 2d_R(b)a^2 + 4d_R(a)ab + 2d_R(aba) &= 2d_R(a)ab + 6d_R(a)ba + 4d_R(b)a^2 \\ 2d_R(aba) &= 2d_R(b)a^2 - 2d_R(a)ab + 6d_R(a)ba \\ d_R(aba) &= d_R(b)a^2 + 3d_R(a)ba - d_R(a)ab. \end{aligned}$$

replace b by $cb + bc$ in (1)

$$\begin{aligned} d_R(a(cb+bc) + (cb+bc)a) &= 2d_R(a)(cb+bc) + 2d_R(cb+bc)a \\ &= 2d_R(a)cb + 2d_R(a)bc + 4d_R(c)ba + 4d_R(b)ca \quad \cdots\cdots(4) \\ d_R(acb + abc + cba + bca) &= d_R(acb) + d_R(abc + cba) + d_R(bca) \\ &= d_R(ac)b + d_R(b)ac + d_R(abc+cba) + d_R(bc)a + d_R(a)bc \\ &= d_R(a)cb + d_R(c)ab + d_R(b)ac + d_R(abc+cba) + \\ &\quad d_R(b)ca + d_R(c)ba + d_R(a)bc \quad \cdots\cdots(5) \end{aligned}$$

From (4) and (5)

$$\begin{aligned} &2d_R(a)cb + 2d_R(a)bc + 4d_R(c)ba + 4d_R(b)ca \\ &\qquad = d_R(a)cb + d_R(c)ab + d_R(b)ac + \\ &\qquad\quad\; d_R(abc+cba) + d_R(b)ca + d_R(c)ba + d_R(a)bc \end{aligned}$$

$$d_R(abc+cba) = d_R(a)(cb+bc) + 3d_R(b)ca + 3d_R(c)ba - d_R(c)ab - d_R(b)ac \quad \cdots\cdots(6)$$

Replace c by ba in (6)

$$\begin{aligned} d_R(abba + baba) &= d_R(a)(bab + bba) + 3d_R(b)baa + \\ &\quad 3d_R(ba)ba - d_R(ba)ab - d_R(b)aba \\ &= d_R(a)(ba + b^2a) + 3d_R(b)ba^2 + \\ &\quad 3d_R(ba)ba - d_R(ba)ab - d_R(b)aba \quad \cdots(7) \\ d_R(ab^2a + (ba)^2) &= d_R(ab^2a) + d_R(ba)^2 \\ &= d_R(ab^2)a + d_R(a)ab^2 + 2d_R(ba)ba \\ &= d_R(a)b^2a + d_R(b^2)a^2 + d_R(a)ab^2 + 2d_R(ba)ba \\ &= d_R(a)b^2a + 2d_R(b)ba^2 + d_R(a)ab^2 + 2d_R(ba)ba \quad \cdots\cdots(8) \end{aligned}$$

From (7) and (8)

$$\begin{aligned} &d_R(a)bab + d_R(a)b^2a + 3d_R(b)ba^2 + 3d_R(ba)ba - d_R(ba)ab - d_R(b)ab \\ &\qquad = d_R(a)b^2a + 2d_R(b)ba^2 + d_R(a)ab^2 + 2d_R(ba)ba \\ d_R(ba)bab - d_R(ba)ab &= d_R(b)aba - d_R(a)bab - d_R(a)b^2a - 3d_R(b)ba^2 + \\ &\quad d_R(a)b^2a + 2d_R(b)ba^2 + d_R(a)ab^2 \\ d_R(ba)(ba - ab) &= d_R(b)(ab - ba)a + d_R(a)(ab - ba)b \quad \cdots\cdots(9) \end{aligned}$$

Replace b by $a+b$ in (9)

$$\begin{aligned} d_R((a+b)a)[(a+b)a-a(a+b)] &= d_R(a+b)[a(a+b)-(a+b)a]a+ \\ & \quad d_R(a)[a(a+b)-(a+b)a](a+b) \\ d_R(a^2+ba)(ba-ab) &= (d_R(a)+d_R(b))(ab-ba)a+d_R(a)(ab-ba)(a+b) \\ (d_R(a^2)+d_R(ba))(ba-ab) &= d_R(a)(ab-ba)a+d_R(b)(ab-ba)a+ \\ & \quad d_R(a)(ab-ba)a+d_R(a)(ab-ba)b \end{aligned}$$

$$2d_R(a)a(ba-ab)+d_R(b)(ab-ba)a+d_R(a)(ab-ba)b$$

$$\begin{aligned} &= 2d_R(a)(ab-ba)a+d_R(b)(ab-ba)a+d_R(a)(ab-ba)b \textit{ using } (9) \\ d_R(a)a(ba-ab) &= d_R(a)(ab-ba)a \quad \cdots\cdots(10) \end{aligned}$$

Replace a by $a+b$ in (10) $d_R(a+b)(a+b)(b(a+b)-(a+b)b) = d_R(a+b)((a+b)b+b(a+b))(a+b)$

$d_R(a+b)(a+b)(ba+b^2-ab-b^2) = [d_R(a)+d_R(b)](ab+b^2-ba-b^2)(a+b)$

$(d_R(a)+d_R(b))(a+b)(ba-ab) \quad = (d_R(a)+d_R(b))(ab-ba)(a+b)$

$d_R(a)a(ba-ab)+d_R(a)b(ba-ab)+d_R(b)a(ba-ab)+d_R(b)b(ba-ab) = d_R(a)(ab-ba)a \quad +d_R(ba)(ba-ab)+d_R(b)(ab-ba)b using (9)$

$d_R(a)b(ba-ab)+d_R(b)a(ba-ab) = d_R(ba)(ba-ab)$

$(ba-ab)[d_R(ab)-d_R(a)b-d_R(b)a] = 0.$

Analogously we can prove the following.

Theorem 3.9 *Let S be a semiring and X be a 2-torsion free left S-module. If $d_L : S \to X$ is a left-derivation then $a,b,c \in R$, then*

1. $d_L(ab+ba) = 2ad_L(b)+2bd_L(a)$.
2. $d_L(aba) = a2d_L(b)+3abd_L(a)-bad_L(a)$.
3. $d_L(abc+cba) = 3abd_L(c)+3acd_L(b)+(bc+cb)d_L(a)-bad_L(c)-cad_L(b)$.
4. $(ab-ba)ad_L(a) = a(ba-ab)d_L(a)$.
5. $(ab-ba)(d_L(ab)-ad_L(b)-bd_L(a)) = 0$.

Theorem 3.10 *Let S be an additively commutative and cancellative semiring. Let X be a right $S-$module. Let $d_R : S \to X$ be a non zero right-derivation. Suppose that $xSa = 0$ with $a \in S, x \in X \;\Rightarrow\; a = 0$ or $x = 0$. Then S is commutative.*

Proof: Let S be an additively commutative and cancellative semiring.

Let $d_R : S \to X$ be a non zero right-derivation on S.

$$\begin{aligned} d_R(aba) &= d_R(a)ba + d_R(ba)a \\ &= d_R(a)ba + d_R(b)a^2 + d_R(a)ba \\ &= 2d_R(a)ba + d_R(b)a^2 \quad \cdots\cdots(1) \end{aligned}$$

$$\begin{aligned} d_R(aba) &= d_R(ab)a + d_R(a)ab \\ &= d_R(a)ba + d_R(b)a^2 + d_R(a)ab \\ &= d_R(a)(ba + ab) + d_R(b)a^2 \quad \cdots\cdots(2) \end{aligned}$$

From (1) and (2) $2d_R(a)ba + d_R(b)a^2 = d_R(a)(ab + ba) + d_R(b)a^2$
$d_R(a)ba = d_R(a)ab$

$$d_R(a)[a, b] = 0 \ \ \forall\ a, b \in S \qquad\qquad \cdots\cdots(3)$$

Replacing b by cb, and using (3) we get

$$d_R(a)c[a, b] = 0 \ \ \forall\ a, b, c \in S \Rightarrow d_R(a)S[a, b] = 0 \ \ \forall\ a, b \ \in\ S.$$

Since $d_R \neq 0$, we get $[a, b] = 0\ \forall\ a, b \in S$, thus proving that S is commutative.

Analogously we can prove the following.

Theorem 3.11 *Let S be an additively commutative and cancellative semiring. Let X be a left $S-$module. Let $d_L : S \to X$ be a non zero left derivation. Then S is commutative.*

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