International Mathematical Forum, Vol. 8, 2013, no. 32, 1561 - 1568
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/imf.2013.38164

A Note Involving Two-by-Two Matrices of the k-Pell and k-Pell-Lucas Sequences

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Abstract

We use a diagonal matrix for getting the Binet's formula for k-Pell sequence Also the n^{th} power of the generating matrix for k-Pell-Lucas sequence is established and basic properties involving the determinant allow us to obtain its Cassini's identity.

Mathematics Subject Classification: 11B37, 05A15, 11B83.

Keywords: Pell numbers, Pell-Lucas numbers, k-Pell numbers, k-Pell-Lucas numbers, Binet's formula, Generating matrix, Matrix diagonalization

1 Introduction

The well-known Fibonacci (and Lucas) sequence is one of the sequences of positive integers that have been studied over several years. Many authors are dedicated to study this sequence, such as the work of Hoggatt, in [19] and Vorobiov, in [12], among others. Several relations of this sequence with different scientific areas can be found in the literature. For example, S. Klavžar [20] uses tools of graph theory and consider the Fibonacci cube with emphasis on their structure and J. Y. Cai *et al.* in [22] introduce Fibonacci gates as a polynomial

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time computable primitive and develop a theory of holographic algorithms based on these gates. Also the relation with algebra, we have the paper of F. Luca [21] and the works of Caldwell and Komatsu, in [4], Marques in [7], Shattuck in [10] and Falcón and Plaza, in [16]. The matrix method is used to get some properties for some sequences of numbers. For example, in [16], the authors consider some properties for the *k*-Fibonacci numbers obtained from elementary matrix algebra and its identities including generating function and divisibility properties appears in the paper of Bolat and Köse, in [3]. The sequence of Pell numbers is other sequence of numbers that is defined by the recursive sequence given by $P_n = 2P_{n-1} + P_{n-2}$, $n \geq 2$, with the initial conditions $P_0 = 0$ and $P_1 = 1$. This sequence has been studied and some of its basic properties are known (see, for example, the study of Horadam, in [2], among others). In [9], we find the matrix method for generating this sequence and comparable matrix generators have been considered by Kalman, in [6], by Bicknell, in [11], for the Fibonacci and Pell sequences. Also in [18], Koshy studies the relation with the Pascal's Triangle and the sequences of Fibonacci, Lucas and Pell numbers. Sometimes, in the literature, are considered other sequences namely, Pell-Lucas and Modified Pell sequences (see, for example, [17]) and also Dasdemir, in [1], consider new matrices which are based on these sequences as well as that they have the generating matrices The Pell-Lucas sequence is defined by $Q_n = 2Q_{n-1} + Q_{n-2}$, $n \geq 2$, with the initial conditions $Q_0 = Q_1 = 2$. Other sequences, the *k*-Pell and the *k*-Pell-Lucas are considered in Catarino [13], Catarino and Vasco [15] and [14]. For any positive real number *k*, the *k*-Pell sequence say $\{P_{k,n}\}_{n \in \mathbb{N}}$ is defined recurrently by

$$P_{k,n+1} = 2P_{k,n} + \mathrm{k}P_{k,n-1}, \text{ for } n \geq 1, \tag{1}$$

with the initial conditions given by $P_{k,0} = 0$, $P_{k,1} = 1$ (see [13] and [15]) and the *k*-Pell-Lucas $\{Q_{k,n}\}_{n \in \mathbb{N}}$ sequence satisfies the recursive recurrence given by

$$Q_{k,n+1} = 2Q_{k,n} + kQ_{k,n-1}, \text{ for } n \geq 1, \tag{2}$$

with the initial conditions given by $Q_{k,0} = 2$, $Q_{k,1} = 2$ (see the work of Catarino and Vasco, in [14]). The Binet's formula is also well known for several of these sequences. Claude Levesque in [5] finds the general Binet's formula for a general n^{th} order linear recurrence and sometimes this formula can be deduced using matrix method. For example, for the sequence of Jacobsthal number, Koken and Bozkurt, in [8], deduce some properties and the Binet's formula, using matrix method.

In this paper we use a diagonal matrix for getting the Binet's formula for *k*-Pell sequence while for the *k*-Pell-Lucas sequence, we adopt the process considered by Catarino and Vasco, in [14]. Also we stated two identities involving the sequences and use them in the last section when we establish the n^{th} power of the generating matrix for *k*-Pell-Lucas sequence. Finally, some basic properties involving the determinant of this matrix allow us to obtain the Cassini's identity for *k*-Pell-Lucas sequence.

2 Matrix diagonalization of generating matrix for the *k*-Pell sequences

Matrix diagonalization is the process of taking a square matrix and converting it into a special type of matrix called diagonal matrix that shares the same fundamental properties of the underlying matrix. To diagonalize a matrix is also equivalent to find the matrix's eigenvalues, which turn out to be precisely the entries of the diagonalized matrix. The remarkable relationship between a diagonalized matrix, eigenvalues, and eigenvectors follows from the mathematical identity that a square matrix A can be decomposed into the very special form $A = PDP^{-1}$, where P is a matrix consisting of the eigenvectors of A in column, D is the diagonal matrix constructed from the corresponding eigenvalues, and P^{-1} is the matrix inverse of P.

Next we use this process in order to get the Binet's formula of the *k*-Pell sequence. Following Levesque in [5], the *k*-Pell sequence is one (among others recursive sequences) of the special cases of a sequence which is defined recursively as a linear combination of the preceding p terms

$$a_{n+p} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{p-1} a_{n+p-1}, \tag{3}$$

where c_0, c_1, . . . , $c_{p-1}$are real constants. Using the matrix method, consider a matrix T of order $p \times p$ such that the last line is consisting of the constants c_0, c_1, . . . , c_{p-1} and the entries $t_{i,i+1} = 1$, for $i = 1, \ldots p-1$ and the remaining entries zero. Also define a matrix $A_n = \left(a_n, \ldots, a_{n+p-1}\right)^T$ associated with (3). It is easy to show that $TA_n = A_{n+1}$ and $A_n = T^n A_0$, where $A_0 = \left(a_0, \ldots, a_{p-1}\right)^T$and results of linear algebra will be used in what is follows in order to give another expression for the general term of the *k*-Pell sequence. Also we shall use the eigenvalues and the respective eigenvectors in this process. If we recall the recurrence (1), then according (3), we have that $p = 2$, $c_0 = k$ and $c_1 = 2$. Hence the matrix associated is given by $T = \begin{pmatrix} 0 & 1 \\ c_0 & c_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ k & 2 \end{pmatrix}$. In section 3 of Catarino and Vasco, in [15], we find this matrix and also the proof of the n^{th} power of T, $T^n = \begin{pmatrix} kP_{k,n-1} & P_{k,n} \\ kP_{k,n} & P_{k,n+1} \end{pmatrix}$, for all $n \geq 1$. Using the diagonalization of T, the eigenvalues of T are $r_1 = 1 + \sqrt{1+k}$ and $r_2 = 1 - \sqrt{1+k}$ and then $D = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$. The eigenvectors associated with r_1 and $r_2$are respectively,$\begin{pmatrix} x \\ r_1 x \end{pmatrix}$, $\begin{pmatrix} x \\ r_2 x \end{pmatrix}$, with x non zero. In particular, for $x = 1$, we get the eigenvectors $v_1 = \begin{pmatrix} 1 \\ r_1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ r_2 \end{pmatrix}$. Then we have $P = \begin{pmatrix} 1 & 1 \\ r_1 & r_2 \end{pmatrix}$, its inverse is given by $P^{-1} = \begin{pmatrix} \frac{r_2}{r_2 - r_1} & -\frac{1}{r_2 - r_1} \\ -\frac{r_1}{r_2 - r_1} & \frac{1}{r_2 - r_1} \end{pmatrix}$ and easily, $T = PDP^{-1}$. Hence T and D are similar and also $T^n = PD^nP^{-1}$. Now using $A_0 = \begin{pmatrix} P_{k,0} \\ P_{k,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we obtain that $A_n = T^nA_0 = \begin{pmatrix} P_{k,n} \\ P_{k,n+1} \end{pmatrix}$ and then we achieve the Binet's formula for the k-Pell sequence.

Proposition 1 (Binet's formula): *The* n^{th} *k-Pell number is given by*

$$P_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}, \tag{4}$$

where, r_1*,* r_2 *are the roots of the characteristic equation* $r^2 - 2r - k = 0$ *associated to the recurrence relation* (1) *and* $r_1 > r_2$.

■

3 Particular identities involving k-Pell and k-Pell-Lucas numbers

In this section, we consider two identities involving both k-Pell and k-Pell-Lucas numbers and we use them in the next section in order to find the n^{th} power of the generating matrix for the k-Pell-Lucas sequence. First, we recall the Binet's formula for k-Pell-Lucas sequence,

Proposition 2 (Binet's formula): [Proposition 1, in Catarino *et al.* [14]] *The* n^{th} *k-Pell-Lucas number is given by*

$$Q_{k,n} = r_1^n + r_2^n, \tag{5}$$

where, r_1*,* r_2 *are the roots of the characteristic equation* $r^2 - 2r - k = 0$ *associated to the recurrence relation* (2) *and* $r_1 > r_2$.

■

Using Binet's formulas (4) and (5), we obtain the first identity involving the sequences referred.

Proposition 3: *For all* $n \geq 0$,

$$Q_{k,n} = 2(P_{k,n+1} - P_{k,n}). \tag{6}$$

Proof: Using (4), we have,
$P_{k,n+1} - P_{k,n} = \frac{r_1^{n+1} - r_2^{n+1} - r_1^n + r_2^n}{r_1 - r_2} = \frac{r_1^n(r_1 - 1) + r_2^n(1 - r_2)}{r_1 - r_2} = \frac{r_1^n(\sqrt{1+k}) + r_2^n(\sqrt{1+k})}{r_1 - r_2} = \frac{\sqrt{1+k}\,(r_1^n + r_2^n)}{2\sqrt{1+k}} = \frac{(r_1^n + r_2^n)}{2}$, and then, the result follows using the formula (5).

■

Also using a similar calculation, we obtain the second identity involving both sequences.

Proposition 4: *For all* $n \geq 0$,

$$Q_{k,n+1} = 2\left(P_{k,n+1} + P_{k,n}\right). \tag{7}$$

Proof: Using (6), we have, $Q_{k,n+1} = 2\left(P_{k,n+2} - P_{k,n+1}\right)$. Now according (1), we obtain that $Q_{k,n+1} = 2\left(2P_{k,n+1} + kP_{k,n}\right) - 2P_{k,n+1} = 2P_{k,n+1} + 2kP_{k,n}$.

■

4 Generating matrix for the *k*-Pell-Lucas sequences

As it is known, one of the most usual and recurrent recent methods for the study of the recurrences sequences is to define the so-called generating matrix and we can find several research in this topic. Next we shall study this problem for the *k*-Pell-Lucas sequences. The *k*-Pell-Lucas sequence is also a sequence which is defined recursively by (3) as a linear combination of the preceding p terms, where $c_0, c_1, \ldots, c_{p-1}$are real constants. Using the matrix method stated in the previous section, if we recall the recurrence (2), then according (3), we have $p = 2$, $c_0 = k$ and $c_1 = 2$. Hence the matrix Q associated is given by $Q = \begin{pmatrix} 0 & 1 \\ c_0 & c_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ k & 2 \end{pmatrix}$, with $|Q| = -k$. Note that $Q = T$, the generating matrix of *k*-Pell sequence. Now, we shall find the entries of Q^nexpressed with some terms of the *k*-Pell-Lucas sequence. Using the identities of previews section, if we subtracting (6)-(7), we obtain that

$$P_{k,n} = \frac{Q_{k,n+1} - Q_{k,n}}{2(k+1)}. \tag{8}$$

Using (8) in the entries of T^n, we can conclude that

Proposition 5:

$$Q^n = \begin{pmatrix} k\left(\frac{Q_{k,n} - Q_{k,n-1}}{2(k+1)}\right) & \frac{Q_{k,n+1} - Q_{k,n}}{2(k+1)} \\ k\left(\frac{Q_{k,n+1} - Q_{k,n}}{2(k+1)}\right) & \frac{Q_{k,n+2} - Q_{k,n+1}}{2(k+1)} \end{pmatrix}, \tag{9}$$

for all $n \geq 1$.

Proof: We shall use induction on n.

For $n = 1$, $Q = \begin{pmatrix} k\left(\frac{Q_{k,1} - Q_{k,0}}{2(k+1)}\right) & \frac{Q_{k,2} - Q_{k,1}}{2(k+1)} \\ k\left(\frac{Q_{k,2} - Q_{k,1}}{2(k+1)}\right) & \frac{Q_{k,3} - Q_{k,2}}{2(k+1)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ k & 2 \end{pmatrix}$, that is true using the initial conditions of sequence. Suppose now that (9) is valid for n and using (2), then we get that

$$Q^{n+1} = Q^n Q = \begin{pmatrix} k\left(\frac{Q_{k,n} - Q_{k,n-1}}{2(k+1)}\right) & \frac{Q_{k,n+1} - Q_{k,n}}{2(k+1)} \\ k\left(\frac{Q_{k,n+1} - Q_{k,n}}{2(k+1)}\right) & \frac{Q_{k,n+2} - Q_{k,n+1}}{2(k+1)} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ k & 2 \end{pmatrix}$$

$$= \begin{pmatrix} k\left(\frac{Q_{k,n+1}-Q_{k,n}}{2(k+1)}\right) & k\left(\frac{Q_{k,n}-Q_{k,n-1}}{2(k+1)}\right)+2\left(\frac{Q_{k,n+1}-Q_{k,n}}{2(k+1)}\right) \\ k\left(\frac{Q_{k,n+2}-Q_{k,n+1}}{2(k+1)}\right) & k\left(\frac{Q_{k,n+1}-Q_{k,n}}{2(k+1)}\right)+2\left(\frac{Q_{k,n+2}-Q_{k,n+1}}{2(k+1)}\right) \end{pmatrix}$$

$$= \begin{pmatrix} k\left(\frac{Q_{k,n+1}-Q_{k,n}}{2(k+1)}\right) & \frac{Q_{k,n+2}-Q_{k,n+1}}{2(k+1)} \\ k\left(\frac{Q_{k,n+1}-Q_{k,n}}{2(k+1)}\right) & \frac{Q_{k,n+3}-Q_{k,n+2}}{2(k+1)} \end{pmatrix},$$

as required. ■

Using the properties involving the determinant of the matrices Q and Q^n and doing some calculations, we can obtain the Cassini's identity.

Proposition 6 (Cassini's identity):

$$Q_{k,n-1}Q_{k,n+1} - Q_{k,n}^2 = 4(-k)^{n-1}(1+k).$$

Proof: In fact, $|Q^n| = |Q|^n$, and so $|Q^n| = (-k)^n$. Next we consider the expression of determinant of $Q^n$and doing some calculations, we have that

$$|Q^n| = \frac{k}{4(k+1)^n}\left(Q_{k,n-1}Q_{k,n+1} - {Q_{k,n}}^2 + A\right) \qquad (10)$$

where

$$A = Q_{k,n}Q_{k,n+2} + Q_{k,n}Q_{k,n+1} - Q_{k,n-1}Q_{k,n+2} - {Q_{k,n+1}}^2. \qquad (11)$$

Now doing some calculations and using (8), the expression (11) obtain the following aspect $A = Q_{k,n+2}\left(Q_{k,n} - Q_{k,n-1}\right) + Q_{k,n+1}\left(Q_{k,n} - Q_{k,n+1}\right) = Q_{k,n+2}\left(2P_{k,n-1}(k+1)\right) + Q_{k,n+1}\left(-2P_{k,n}(k+1)\right)$. Now using (4) and (5), with some calculations and simplifications, we get $A = 4k(k+1)(-k)^{n-1}$. Using (10) and the fact that $|Q^n| = (-k)^n$, we have

$$(-k)^n = \frac{k}{4(k+1)^n}\left(Q_{k,n-1}Q_{k,n+1} - {Q_{k,n}}^2 + 4k(k+1)(-k)^{n-1}\right),$$

and hence $Q_{k,n-1}Q_{k,n+1} - {Q_{k,n}}^2 = 4(-k)^{n-1}(1+k)$, as required.

■

The eigenvalues of Q coincides with $r_1$and r_2 and the eigenvectors associated with r_1 and $r_2$are respectively,$\begin{pmatrix} x \\ r_1x \end{pmatrix}$, $\begin{pmatrix} x \\ r_2x \end{pmatrix}$, with x non zero. In particular, for $x = 1$, we get the eigenvectors $v_1 = \begin{pmatrix} 1 \\ r_1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ r_2 \end{pmatrix}$. Writing, in this case, $\begin{pmatrix} Q_{k,0} \\ Q_{k,1} \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \alpha_1 v_1 + \alpha_2 v_2$, we obtain that $\alpha_1 = \frac{2r_2-2}{r_1-r_2}$ and $\alpha_2 = -\frac{2-2r_1}{r_1-r_2}$. Finally, applying$Q^n$, we get $A_n = Q^n\begin{pmatrix} Q_{k,0} \\ Q_{k,1} \end{pmatrix} = \begin{pmatrix} Q_{k,n} \\ Q_{k,n+1} \end{pmatrix}$ and then we achieve the Binet's formula for the k-Pell-Lucas sequence, obtained in different way that Catarino and Vasco did in [14].

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Received: August 27, 2013