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Axioms of Countability in Generalized Topological Spaces

John Benedict T. Ayawan [1]

Sciences Cluster
University of the Philippines Cebu
6000 Cebu City, Philippines
jbenz321@gmail.com

Sergio R. Canoy, Jr.

Department of Mathematics
College of Science and Mathematics, MSU-IIT
9200 Iligan City, Philippines
serge_canoy@yahoo.com

Abstract

The different axioms of countability are generalized in such a way that topologies are replaced by generalized topologies in the sense of [1]. In particular, the concepts such as μ-first countable, μ-second countable, and μ-separable are defined, where μ is a generalized topology on a nonempty set X.

Keywords: generalized topological space, μ-first countable, μ-second countable, μ-separable

1 Introduction

A generalization of the concept of topology is that of generalized topology defined in [2]. Specifically, a subset μ of the power set $\exp X$ of X is a *generalized*

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topology (briefly GT) on X if $\emptyset \in \mu$ and every union of elements of μ belongs to μ. From this definition, it follows that every topology on X is a generalized topology. The elements of a generalized topology μ on X are called μ-*open* sets. The union of all the elements of μ is denoted by M_μ. A generalized topology is said to be *strong* if $M_\mu = X$.

The purpose of this paper is to define first countability, second countability, and separability in generalized topological spaces. We consider some properties of these concepts and characterize first countability and second countability of the product of GT's.

2 Preliminaries

Let (X, μ) be a generalized topological space. We say that $M \subset X$ is μ-*open* if $M \in \mu$; $N \subset X$ is μ-*closed* if $X - N \in \mu$. If $A \subseteq X$, then $i_\mu A$ is the union of all μ-open sets contained in A and $c_\mu A$ is the intersection of all μ-closed sets containing A (see [3]). It is known that $c_\mu A \subseteq c_\mu B$, whenever $A \subseteq B$ (see [5]). If $c_\mu A = X$, then A is said to be μ-*dense* in X [4]. Let $\mathrm{B} \subseteq \exp X$ and $\emptyset \in \mathrm{B}$. Then B is a *base* for μ if $\{ \bigcup_{B^* \in \mathrm{B}_1} B^* : \mathrm{B}_1 \subseteq \mathrm{B}\} = \mu$ [6]. We also say that μ is generated by B. It is shown in [5] that B is a base for μ iff whenever U is a μ-open set and $x \in U$, there exists $B \in \mathrm{B}$ such that $x \in B \subseteq U$. A class B_p of μ-open sets containing p is called a μ-local base at p, if for each μ-open set U containing p, there is $U_p \in \mathrm{B}_p$ with $p \in U_p \subseteq U$.

3 μ-first countable, μ-second countable and μ-separable spaces

Definition 3.1 A GTS (X, μ) is called
(1) a μ-first countable space if there is a countable μ-local base at every $p \in M_\mu$.
(2) a μ-second countable space if there is a countable base for the generalized topology μ.
(3) a μ-separable space if X contains a countable dense subset.

Example 3.2 Let $X = \mathbb{R}$ and let $\mu = \mathrm{A}_1 \cup \mathrm{A}_2 \cup \{\emptyset, \mathbb{R}\}$ where,

$$\begin{aligned} \mathrm{A}_1 &= \{(-\infty, a_1] : a_1 \geq 2\} \cup \{(-\infty, a_2) : a_2 \geq 2\} \text{ and} \\ \mathrm{A}_2 &= \{[b_1, +\infty) : b_1 \leq 1\} \cup \{(b_2, +\infty) : b_2 \leq 1\}. \end{aligned}$$

By the Completeness Axiom, it is easy to show that $(\mathbb{R}, \mu)$ is a GTS. Observe that since $(-\infty, 3] \cap [a, +\infty) = [1, 3] \notin \mu$, $(\mathbb{R}, \mu)$ is not a topological space.

Now, for each $x \in M_\mu = \mathbb{R}$, consider the following families:

$$\begin{aligned} \mathrm{B}_x &= \{(-\infty, x], (1, +\infty)\} \text{ if } x \geq 2, \\ \mathrm{B}_x &= \{[x, +\infty), (-\infty, 2)\} \text{ if } x \leq 1 \text{ and} \\ \mathrm{B}_x &= \{(-\infty, 2), (1, +\infty)\} \text{ if } 1 < x < 2. \end{aligned}$$

It can be be verified that for each case, B_x is μ-local base at x. Thus $(\mathbb{R}, \mu)$ is a μ-first countable space. Also, since $\mathbb{Q}$ is a countable μ-dense subset of $\mathbb{R}$, $(\mathbb{R}, \mu)$ is a μ-separable space.

Example 3.3 Let $X = \mathbb{R}$ and let $\mu = \mathrm{A}_1 \cup \mathrm{A}_2 \cup \{\emptyset, \mathbb{R}\}$, where

$$\begin{aligned} \mathrm{A}_1 &= \{(b, +\infty) : b \leq 1\} \text{ and} \\ \mathrm{A}_2 &= \{(-\infty, a) : a \geq 2\}. \end{aligned}$$

Again, the Completeness Axiom can be used to show that $(\mathbb{R}, \mu)$ is a GTS. Moreover, the collection $\mathrm{B} = \{\emptyset\} \cup \{(q_1, +\infty) : q_1 \in \mathbb{Q} \text{ and } q_1 \leq 1\} \cup \{(-\infty, q_2) : q_2 \in \mathbb{Q} \text{ and } q_2 \geq 2)\}$ is a countable base for the GTS $(\mathbb{R}, \mu)$. Therefore, $(\mathbb{R}, \mu)$ is a μ-second countable space.

Theorem 3.4 *Let* B *be a base for a generalized topology* μ *in* X *and* $p \in M_\mu$. *Then* $\mathrm{B}_p = \{B \in \mathrm{B} : p \in B\}$ *is a* μ*-local base at* p.

Proof. Suppose that B is a base for a generalized topology μ in X. Let $U \in \mu$ and $p \in U$. Set $\mathrm{B}_p = \{B \in \mathrm{B} : p \in B\}$. Since B is a base, there exists a subclass B^* of B such that $U = \bigcup_{B \in \mathrm{B}^*} B$. Since $p \in U$, there exists B such that $p \in B$. Thus, $B \in \mathrm{B}_p$ and $p \in B \subseteq U$. Therefore, B_p is a μ-local base at p. ■

Theorem 3.5 *The following statements are equivalent :*
(i) D *is* μ*-dense in* X.
(ii) If F *is* μ*-closed and* $D \subseteq F$, *then* $F = X$.
(iii) If B *is a base for* μ *and* $B \in \mathrm{B}$, *where* $B \neq \emptyset$, *then* $B \cap D \neq \emptyset$.

Proof. $(i) \Rightarrow (ii)$ Note that $D \subseteq F$ implies $c_\mu(D) \subseteq c_\mu(F)$. Since D is μ-dense in X, $X = c_\mu(D) \subseteq c_\mu(F) = F$.
$(ii) \Rightarrow (iii)$ Suppose that $B \cap D = \emptyset$ for some nonempty $B \in \mathrm{B}$. Then $D \subseteq B^c$. Since B^c is μ-closed, $B^c = X$, by assumption. This implies that $B = \emptyset$, contrary to the assumption of B.
$(iii) \Rightarrow (i)$ Suppose that D is not μ-dense in X. This means that $c_\mu(D) = T = \cap\{F : F \text{ is } \mu\text{-closed with } D \subseteq F\} \neq X$. Hence, there exists $x \in X \setminus T$. This implies that there exists a μ-closed set F with $D \subseteq F$

such that $x \notin F$. Since F^c is a μ-open set and $x \in F^c$, there exists $B \in$ B such that $x \in B \subseteq F^c$. Consequently, $B \cap D \subseteq F^c \cap D$. Since $F^c \cap D = \emptyset$, it follows that $B \cap D = \emptyset$. This contradicts our assumption. Therefore, D is μ-dense in X. ■

Theorem 3.6 *Every μ-second countable generalized topological space is μ-first countable.*

Proof. Suppose that X is μ-second countable. Then X has a countable base B. Let $p \in M_\mu$. By Theorem 3.4, $\mathrm{B}_p = \{B \in \mathrm{B} : p \in B\}$ is a μ-local base at p. Since B is countable and $\mathrm{B}_p \subseteq \mathrm{B}$ it follows that B_p is a countable μ-local base at p. This shows that X is a μ-first countable space. ■

Remark 3.7 *The converse of Theorem 3.6 is not true.*

To see this, consider the μ-first countable GTS $(\mathbb{R}, \mu)$ in Example 3.2. Let $\mathrm{S} = \{[b, +\infty) : b \leq 1\} \cup \{(-\infty, a] : a \geq 2\}$ and let B be a base for μ. Then S is uncountable. Note that $b \in [b, +\infty)$ and the only μ-open set containing b which is contained in $[b, +\infty)$ is the set itself. Hence, $[b, +\infty) \in$ B for each $b \in \mathbb{R}$ with $b \leq 1$. Similarly, $(-\infty, a] \in$ B for each $a \in \mathbb{R}$ with $a \geq 2$. Thus, $\mathrm{B} \supseteq \mathrm{S}$. Therefore, B is uncountable. This shows that (X, μ) is not μ-second countable.

Theorem 3.8 *Every μ-second countable generalized topological space is μ-separable.*

Proof. Let (X, μ) be μ-second countable space and let $\mathrm{B} = \{B_n : n \in \mathbb{N}\}$ be a countable base for μ. Assume that $B_1 = \emptyset$. For each $n \in \mathbb{N} \setminus \{1\}$ choose a point $a_n \in B_n$. The set $A = \{a_n : n \in \mathbb{N} \setminus \{1\}\}$ is also countable. Note that $A \cap B_n \neq \emptyset$ since $a_n \in A \cap B_n$ for all $n \in \mathbb{N} \setminus \{1\}$. By Theorem 3.5, A is μ-dense in X. Therefore, (X, μ) is μ-separable. ■

$(\mathbb{R}, \mu)$ in Example 3.2 is μ-separable but not μ-second countable by Remark 3.7. From this, the following remark is immediate.

Remark 3.9 *The converse of Theorem 3.8 is not true.*

Remark 3.10 *μ-first countability does not imply μ-separability.*

Consider $X = \mathbb{R}$, $\mu = \{A : A \subseteq \mathbb{Q}^c\}$. Clearly, μ is a generalized topology but not a topology for $X = \mathbb{R}$ since $\mathbb{R} \notin \mu$. Let $p \in \mathbb{Q}^c$. Then $\{p\} \in \mu$. Clearly, $\{\{p\}\}$ is μ-local base at p. Hence, every $p \in \mathbb{Q}^c = M_\mu$ has a finite μ-local base at p. Thus, (X, μ) is μ-first countable. Let D be a μ-dense set in (X, μ). Then $D \cap A \neq \varnothing \ \forall \ A \in \mu$. Let $p \in \mathbb{Q}^c$. Since $\{p\} \in \mu$, $\{p\} \cap D \neq \varnothing$, i.e., $p \in D$. Thus, $\mathbb{Q}^c \subseteq D$. Since $\mathbb{Q}^c$ is uncountable, it follows that D is uncountable. This shows that (X, μ) is not separable.

Remark 3.11 *μ-separability does not imply μ-first countability.*

To see this, let $X = \mathbb{R}$, and let μ be the generalized topology generated by the base $B = \{(r, r+2) : r \in \mathbb{R})\} \cup \{\varnothing\}$. By density property of $\mathbb{Q}$, $(r, r+2) \cap \mathbb{Q} \neq \varnothing$ for each $r \in \mathbb{R}$. By Theorem 3.5, $\mathbb{Q}$ is μ-dense in $\mathbb{R}$. Thus, $(\mathbb{R}, \mu)$ is μ-separable. Next let B_p be a μ-local base at $p \in \mathbb{R}$. Consider $(r, r+2)$, where $r < p < r+2$. Then $(r, r+2)$ is a μ-open set containing p. Since B_p is a μ-local base at p, there exists $B \in \mathrm{B}_p$ such that $p \in B \subseteq (r, r+2)$. Now, since the only μ-open set B satisfying this is $(r, r+2)$ itself, $B = (r, r+2)$. It follows that $\{(r, r+2) : r < p < r+2\} \subseteq \mathrm{B}_p$. Since $\{(r, r+2) : r < p < r+2\} = \{(r, r+2) : r \in (p-2, p)\}$ is uncountable, B_p is uncountable. Therefore, $(\mathbb{R}, \mu)$ is not μ-first countable.

4 Product of Generalized Topologies

Let $K \neq \emptyset$ be an index set and (X_k, μ_k) $(k \in K)$ a class of GTS's and$X = \prod_{k \in K} X_k$ the cartesian product of the sets X_k. Let B be the collection of all sets of the form $\prod_{k \in K} M_k$, where $M_k \in \mu_k$ and $M_k = M_{\mu_k}$ for all but a finite number of indices k. We call $\mu = \mu(\mathrm{B})$ having B as a (defining) base the *product* of the GT's μ_k (see [3]). The GTS (X, μ) is called the product of the GTS's (X, μ_k).

The following results play important roles in the theorems that follow.

Proposition 4.1 [3] *Let $\{(X_k, \mu_k) : k \in I\}$ be a collection of generalized topological spaces, (X, μ) the product of these GTS's and $A = \prod_{k \in K} A_k$, where $A_k \subseteq X_k$. Then $c_\mu(A) = \prod_{k \in K} c_{\mu_k}(A_k)$.*

Proposition 4.2 [3] *Let $\{(X_i, \mu_i) : i \in I\}$ be a collection of generalized topological spaces and let (X, μ) be the product of these GTS's. Then for each fixed $j \in I$, the projection $p_j : (X, \mu) \to (X_j, \mu_j)$ is (μ, μ_j)-open.*

Proposition 4.3 [3] *Let (X, μ) be the product of the GTS's (X_i, μ_i), where $i \in I$. If each μ_i is strong, then for each fixed $j \in I$, the projection $p_j : (X, \mu) \to (X_j, \mu_j)$ is (μ, μ_j)-continuous.*

Theorem 4.4 *Let $\{(X_i, \mu_i) : i \in I\}$ be a collection of strong generalized topological spaces and let (X, μ) be the product of these GTS's. Then (X, μ) is μ-first countable if and only if each (X_i, μ_i) is μ_i-first countable and for all but countably many indices i, $\mu_i = \{\emptyset, X_i\}$.*

Proof. Suppose that μ_i is strong for each $i \in I$ and (X, μ) is μ-first countable. Let (X_j, μ_j) be one of the component spaces and let x_j be an arbitrary element of X_j. Then there is an $x \in X$ such that $p_j(x) = x_j$, where p_j is the jth projection from X to X_j. By assumption, there is a countable μ-local base at x, say $\{U_n : n \in \mathbb{N}\}$. Observe that each $p_j(U_n)$ is μ_j-open by Proposition 4.2. Now, let $x_j \in V \in \mu_j$. Then $p_j^{-1}(V)$ is a μ-open set containing x since p_j is (μ, μ_j)-continuous by Proposition 4.3. Since the $U_n's$ form a μ-local base at x, there is an $n \in \mathbb{N}$ such that $x \in U_n \subseteq p_j^{-1}(V)$. This implies that $x_j \in p_j(U_n) \subseteq V$. So the collection $\{p_j(U_n) : n \in \mathbb{N}\}$ is a μ_j-local base at x_j. This shows that each X_j is μ_j-first countable.

Next, assume that $J = \{j \in I : \mu_j \neq \{\emptyset, X_j\}\}$ is uncountable. For each $j \in J$, choose a μ_j-open set $U_j \neq X_j$ and $x_j \in U_j$. Then $\{U_j : j \in J\}$ is uncountable. For each $i \in I \setminus J$, pick $y_i \in X_i$. Let $x = \langle z_i \rangle_{i \in I} \in X$, where $z_i = x_i$ for $i \in J$ and $z_i = y_i$ for each $i \in I \setminus J$. Since X is μ-first countable and $x \in M_\mu = X$ (since each μ_i is strong), there is a countable μ-local base $\mathrm{B}_x = \{V_n : n \in \mathbb{N}\}$ at x. Let B be the defining base for μ. For each $n \in \mathbb{N}$, there exists $W_n \in \mathrm{B}$ such that $W_n \subseteq V_n$. Hence, $p_j(W_n) \subseteq p_j(V_n)$ for all $j \in I$ and for each $n \in \mathbb{N}$. Since $p_j(W_n) = X_j$ for all but finitely many $j \in I$, it follows that $p_j(V_n) = X_j$ for all but finitely many $j \in I$ and for each $n \in \mathbb{N}$. For any particular $n \in \mathbb{N}$, let $I_n = \{i \in I : p_i(V_n) \neq X_i\}$. Then I_n is finite for all $n \in \mathbb{N}$. Let $S = \bigcup_{n \in \mathbb{N}} I_n$. Then $S \subseteq I$ and is countable. Since J is uncountable, there exists $k \in J \setminus S$ such that $x_k \in U_k \neq X_k$. Now, $p_k^{-1}(U_k)$ is a μ-open set by Proposition 4.3 and $x \in p_k^{-1}(U_k)$. Since $k \notin S$, $p_k(V_n) = X_k$ for all $n \in \mathbb{N}$. This implies that there exists no $V_n \in \mathrm{B}_x$ such that $x_k \in p_k(V_n) = X_k \subseteq U_k$. Hence, B_x is not a μ-local base at x, contrary to our assumption. Therefore, $\mu_j \neq \{\emptyset, X_j\}$ for countably many $j \in I$.

For the converse, suppose that each X_i is μ_i-first countable and that $\mu_i = \{\emptyset, X_i\}$ for all but countably many indices $i \in I$. Let $C = \{i \in I : \mu_i \neq \{\emptyset, X_i\}\}$. Then C is countable and $X = \prod_{i \in C} X_i \times \prod_{i \in I \setminus C} X_i$. Let $x = \langle x_i \rangle_{i \in I} \in M_\mu = X$. For each $i \in C$, let $\mathrm{B}_{x_i} = \{U_{i,n} : n \in \mathbb{N}\}$ be a countable μ_i-local base at x_i. For any finite subset of $C \times \mathbb{N}$ of the form $F = \{(i_1, k_1), (i_2, k_2), ..., (i_m, k_m)\}$, define $V_F = \prod_{j=1}^{m} M_{i_j,k_j} \times \prod_{i \in I \setminus T} X_i$, where $T = \{i_1, i_2, ..., i_m\}$ and $M_{i_j,k_j} = U_{i_j,k_j} \in \mu_{i_j}$ for each $i_j \in T$. Set $\mathrm{B}_x = \{V_F : F \text{ is a finite subset of } C \times \mathbb{N}\}$. Then V_F is μ-open and $x \in V_F$ for each finite subset F of $C \times \mathbb{N}$. Let $U \in \mu$ with $x \in U$. Then there exists $B \in \mathrm{B}$, where B is the defining base for μ, such that $x \in B \subseteq U$. Then $B = \prod_{j \in J} M_j \times \prod_{i \in I \setminus J} X_i$, where J is a finite subset of C and $M_j = V_j \in \mu_j$ for each $j \in J$. For each $j \in J$, $x_j = p_j(x) \in p_j(B) = V_j$. Since

B_{x_j} is a μ_j-local base at x_j, there exists $U_{j,k_j} \in \mathrm{B}_{x_j}$ such that $x_j \in U_{j,k_j} \subseteq V_j$. Let $V = \prod_{j \in J} U_{j,k_j} \times \prod_{i \in I \setminus J} X_i$. Then $V \in \mathrm{B}_x$ and $x \in V \subseteq B \subseteq U$. This implies that B_x is a countable μ-local base at x. Therefore, X is μ-first countable. ■

Theorem 4.5 *Let $\{(X_i, \mu_i) : i \in I\}$ be a collection of strong generalized topological spaces and let (X, μ) be the product of these GTS's. Then (X, μ) is μ-second countable if and only if each (X_i, μ_i) is μ_i-second countable and for all but countably many indices i, $\mu_i = \{\emptyset, X_i\}$.*

Proof. Suppose that μ_i is strong for each $i \in I$ and (X, μ) is μ-second countable. Let (X_j, μ_j) be one of the component spaces. By assumption, there is a countable base for μ, say $\mathrm{B}^* = \{B_n : n \in \mathbb{N}\}$. Note that each $p_j(B_n)$ is μ_j-open by Proposition 4.2. Let U_j be a μ_j-open set and $x_j \in U_j$. Since p_j is onto there is an $x \in X$ such that $p_j(x) = x_j$, where p_j is the jth projection map from X to X_j. By Proposition 4.3, $p_j^{-1}(U_j)$ is a μ-open set with $x \in p_j^{-1}(U_j)$. Since B^* is a base for μ there exists $n \in \mathbb{N}$ such that $x \in B_n \subseteq p_j^{-1}(U_j)$. This implies that $x_j \in p_j(B_n) \subseteq U_j$. So the collection $\{p_j(B_n) : n \in \mathbb{N}\}$ is a countable base for μ_j This shows that each X_j is μ_j-second countable.

Next, assume that $J = \{j \in I : \mu_j \neq \{\emptyset, X_j\}\}$ is uncountable. Since $B_n \in \mathrm{B}^*$ for all $n \in \mathbb{N}$, $p_j(B_n) = X_j$ for all but finitely many $j \in I$ and for each $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, let $I_n = \{i \in I : p_i(B_n) \neq X_i\}$. Then I_n is finite for all $n \in \mathbb{N}$. Let $S = \bigcup_{n \in \mathbb{N}} I_n$. Then $S \subseteq I$ and is countable. Since J is uncountable, there exists $k \in J \setminus S$ such that $x_k \in U_k \neq X_k$. Now $p_j^{-1}(U_k)$ is a μ-open set by Proposition 4.3 and $x \in p_j^{-1}(U_k)$. Since $k \notin S$, $p_k(B_n) = X_k$ for all $n \in \mathbb{N}$. This implies that there exists no $B_n \in \mathrm{B}$ such that $x_k \in p_k(B_n) = X_k \subseteq U_k$. Hence, B^* is not a base for μ, contrary to our assumption. Therefore, $\mu_j \neq \{\emptyset, X_j\}$ for countably many $j \in I$.

For the converse, suppose that each X_i is μ_i-second countable and that $\mu_i = \{\emptyset, X_i\}$ for all but countably many $i \in I$. Let $C = \{i \in I : \mu_i \neq \{\emptyset, X_i\}\}$. Then C is countable and $X = \prod_{i \in C} X_i \times \prod_{i \in I \setminus C} X_i$. For each $i \in C$, let $\mathrm{B}_i = \{U_{i,n} : n \in \mathbb{N}\}$ be a countable base for μ_i. For any finite subset of $C \times \mathbb{N}$ of the form $F = \{(i_1, k_1), (i_2, k_2), ..., (i_m, k_m)\}$, define $V_F = \prod_{j=1}^{m} M_{i_j,k_j} \times \prod_{i \in I \setminus T} X_i$, where $T = \{i_1, i_2, ..., i_m\}$ and $M_{i_j,k_j} = U_{i_j,k_j} \in \mu_{i_j}$ for each $i_j \in T$. Set $\mathrm{B}^* = \{V_F : F \text{ is a finite subset of } C \times \mathbb{N}\}$. Let $U \in \mu$ and $x \in U$. Then there exists $B \in \mathrm{B}$, where B is the defining base for μ, such that $x \in B \subseteq U$. Then $B = \prod_{j \in J} M_j \times \prod_{i \in I \setminus J} X_i$, where J is a finite subset of C and $M_j = V_j \in \mu_j$ for each $j \in J$. For each $j \in J$, $x_j = p_j(x) \in p_j(B) = V_j$. Since B_j is base for μ_j , there exists $U_{j,k_j} \in \mathrm{B}_j$ such that $x_j \in U_{j,k_j} \subseteq V_j$. Let $V = \prod_{j\in J} U_{j,k_j} \times \prod_{i\in I\setminus J} X_i$. Then $V \in \mathrm{B}^*$ and $x \in V \subseteq B \subseteq U$. This implies that B^* is a countable base for μ. Therefore, X is μ-second countable. ■

Theorem 4.6 *Let $\{(X_i, \mu_i) : i \in I\}$ be a collection of strong generalized topological spaces and let (X, μ) be the product of these GTS's. Then (X, μ) is μ-separable if and only if each (X_j, μ_j) is μ_j-separable.*

Proof. Let D be a countable μ-dense subset of X and B_j be a nonempty member of a base B_j for μ_j. By Proposition 4.3 and the fact that p_j is onto, $p_j^{-1}(B_j)$ is a non-empty μ-open set. Hence $p_j^{-1}(B_j) \cap D \neq \emptyset$ by Theorem 3.5. Thus,

$$\emptyset \neq p_j(p_j^{-1}(B_j) \cap D) \subseteq p_j(p_j^{-1}(B_j)) \cap p_j(D) = B_j \cap p_j(D).$$

This shows that $p_j(D)$ is μ_j-dense in X_j. Also, since D is countable $p_j(D)$ is countable. Therefore, X_j is μ_j-separable.

Suppose that each X_j is μ_j-separable. For each $j \in J$, let D_j be a countable μ_j-dense subset of X_j. Then $\prod_{j\in J} D_j$ is countable. By Proposition 4.1, $c_\mu(\prod_{j\in J} D_j) = \prod_{j\in J} c_\mu(D_j) = \prod_{j\in J} X_j = X$. Therefore, X is μ-separable. ■

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