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The Computation of the Rank of Diagram Groups Constructed from Semigroup Presentation

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Abstract

In this paper, we continue our study on the result of Kilibarda, "the diagram group $D(S,w)$ is isomorphic to the fundamental group $\pi_1(K(S),w)$ of the $2-$complex $K(S)$ with the basepoint w, where S is a monoid presentation and w is positive word over alphabet of S" to construct of the first complex $K(N)$ of diagram group from the semigroup presentation of natural numbers with repeating generators. We will show the diagram group with positive word $w \geq 2$ is free of rank $\mu(n,m) = n^m\left[(m\cdot(n-1)-2)/2 \right]+1$, where n the numbers of generators in $D(N,w)$, $n \geq 2, m \geq 1$ and $n,m \in Z^+$.

Keywords: Diagram groups, semigroup presentation, Squier complex

1. Introduction

Consider set of natural numbers with binary operation + as a semigroup. It can be presented by $[x]$. If we repeat generator, then we obtain repeating generators presentation $[x_i \,/\, x_i = x_j]\ (i \neq j), i = 1,2,\ldots,n$. This presentation is the generalization of our works in [2,3]. As a semigroup presentation, we may construct the diagram group as described by Ahmad [1], Guba and Sapir [2], Kilibarda [3] or by Pride [4]. The diagram group of any semigroup presentation $S = [\boldsymbol{x} : \boldsymbol{r}]$ will be denoted by $D(S,w)$ where w is a positive word on $\boldsymbol{x}$. This group can be obtained from the Squier Complex $K(S)$. In fact the fundamental group of $K(S)$ with the basepoint w denoted by $\pi_1(K(S),w)$ is isomorphic to $D(S,w)$. We will prove

Theorem 1
Let < N, + > be the semigroup of natural number presented by:
$[x_i \,/\, x_i = x_j]\ (i \neq j), i = 1,2,\ldots,n.$
If a word w has length 2($\ell(w) = 2$). Then $D(N,w)$ is a free group of rank $n^3 - 2n^2 + 1\ (\forall \geq 2)$.
Using theorem 1, we will prove by using induction:

Theorem 2
Let $\langle \mathbf{N}, + \rangle$ be the semigroup of natural number presented by:
$[x_i \,/\, x_i = x_j]\ (i \neq j), i = 1,2,\ldots,n.$
If a word w has length m ($\ell(w) = m$). Then the diagram group $D(N,w)$ is a free group of rank $\mu(n,m) = n^m \left[\left(m \cdot (n-1) - 2 \right) \big/ 2 \right] + 1$, where n the numbers of generators in $D(N,w)$ and $n \geq 2, m \geq 1$ and $n, m \in \mathbb{Z}^+$.

2. The Proof Theorem 1

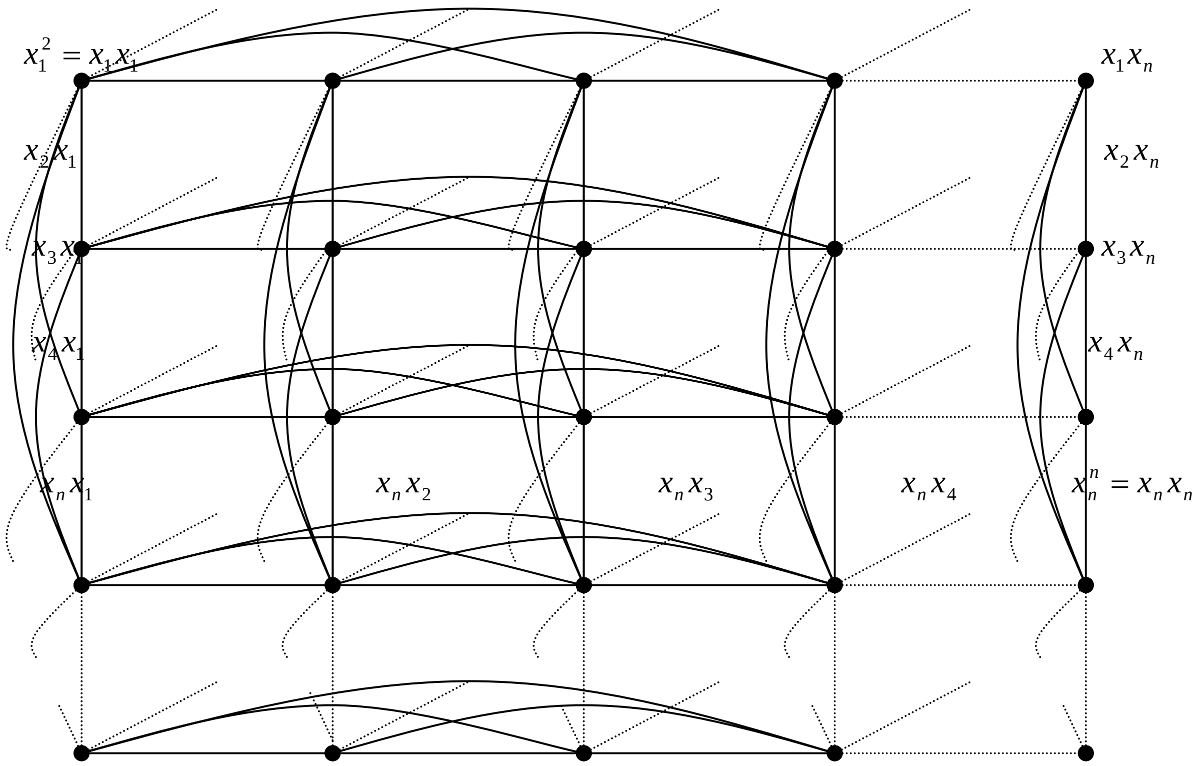
Note that in the Squier complexes $K(N)$, every vertex of length two are connected as in our previous paper in [2] and [3]. Since $K(N)$ has none of 2-cell appeared in it, then it is just a graph. Hence the fundamental group $\pi_1(K(N), w)$ is free and so does $D(N, w)$ .We will prove its rank by using induction. We claim that the function:

$\theta(n) = n^3 - 2n^2 + 1 \ (\forall n \geq 2)$.

Consider all positive words on $K(N)$ with length tow. It's vertices are words as the form elements of matrix $\mathrm{A}_{n\times n} = [x_i x_j]$, $1 \leq i, j \leq n$ (n the order of generators on $S(N)$) where i the index of the row and j the index of the column. Consider the function $\varphi : \mathrm{A} \to \mathrm{A}$ such That:

$$\varphi(x_i x_j) = \begin{cases} x_i x_j, i \neq j \\ x_i^2, i = j \end{cases}$$

to represents the words as elements of matrix A .The vertices (=words) $x_i x_j$ are connected by $(n^2 - n/2)$ relations (edges), the connected components of $x_i x_j$ looks like:



When $k = 2$, we show that the Squier complex is simply a square and hence the fundamental group if free of rank 1. Now suppose that $k > 2$ and $D(N, w)$ is free group of rank $k^3 - 2k^2 + 1$. Note that for each vertex $x_i x_j$ has valence $2(k-1)$ in $K_k(N)$. It is not hard to show that for each vertex $x_i x_j$, it has valence $2k$ in Squier complex $K_{k+1}(N)$. In fact $K_{k+1}(N)$ is the last symbol of words as in the matrix form of component $K_k(N)$, multiplied from the right by new generator x_i in $K_{k+1}(N)$ such that $x_i w$ is for first row and $x_{i+1} w$ for the second row and so on for each $i = 1,2,...,k+1$. Choose the connection between $x_i x_j$ for all i and j as a maximal tree, then there are $k^3 - 2k^2 + 1$ free generators from our assumption . Hence we may conclude that there are $(k+1)^3 - 2(k+1)^2 + 1$ free generator in $K_{k+1}(N)$. It is easy to show that
$(k^3 - 2k^2 + 1) + (3k^2 - k - 1) = (k+1)^3 - 2(k+1)^2 + 1$. For complete the proof.

3. The Proof Theorem 2

Every vertex of length m in the Squire complexes $K(N)$ are connected as in our previous paper in [2, 3]. Since $K(N)$ has none of 2-cell appeared in it, and then it is a graph. Hence the fundamental group $\pi_1(K(N), w)$ is free and so on does $D(N, w)$. We will prove its rank using induction. Let the claim be the function:
$\mu(n,m) = n^m \left[\left(m \cdot (n-1) - 2 \right) / 2 \right] + 1$, where n is the numbers of generators in $D(N, w)$, $n \geq 2, m \geq 1$ and $n, m \in Z^+$. Let b $w = x_1 x_2 ... x_m$ e any positive word on $K(N)$ of length m, then its vertices are words in form elements of matrix $A_{n \times n^2} = [w]$, where n is the number of rows and n^2 is the numbers of columns in the matrix. The numbers of vertices (= words) on the diagram group of natural numbers is given by the following function: $\theta(n,m) = n^m$ and the number of edges (= relations) connected between the words in diagram group is give by the function: $\psi(n,m) = \{m \cdot n^{(m-1)}[n^2 - n]\}/2$. Let $\ell(w) = m = 1$, then the connected

components of $K(N)$ will look like as in our paper in [2] and then $D(N,w)$ is free group of rank: $\mu(n,1) = n^1\left[(^{1\cdot(n-1)-2}\!\diagup_2 \right] + 1 = 1/2\,[n^2 - 3n] + 1$.

If $\ell(w) = m = 2$ then the connected components of $K(N)$ will look like as theorem1 and then $D(N,w)$ is free group of rank: $\eta(n) = n^3 - 2n^2 + 1 (\forall\, n \geq 2)$. Now suppose that $\ell(w) = m = k > 2$ and $D(N,w)$ is free group of rank $\mu(n,k) = n^k\left[(^{k\cdot(n-1)-2}\!\diagup_2 \right] + 1$. Note that for each word w in $K_k(N)$ has valence determined by the function $\beta(n,k) = k(n-1)$. It is not hard to show that for each word w in $K_{k+1}(N)$ has valence determined by the function $\beta(n,k+1) = (k+1)(n-1)$. In fact the Squire Complexes $K_{k+1}(N)$ is just $n-$copies of diagram graph $K_k(N)$. Let $\Delta_i\ (1 \leq i \leq n)$ denoted to the copy of graphs such that $\Delta = x_1\ \Delta_1, x_2\Delta_2, \ldots, x_n\Delta_n$ where $x_i\ (1 \leq i \leq n)$ are the generators in $K_k(N)$, is a new diagram graph represent $K_{k+1}(N)$. Note that the connection between all words with respect to Δ is as follows: the first word in the first row on $\Delta'_1 := x_1\Delta_1$ connects with all first words in the first rows for $\Delta'_2 = x_2\Delta_2, \ldots, \Delta'_n = x_n\Delta_n$ in $K_{k+1}(N)$ and so on for all $2^{nd}, 3^{rd}, \ldots, n^{th}$ words in the first rows in Δ'_1 with $\Delta'_2 = x_2\Delta_2, \ldots, \Delta'_n = x_n\Delta_n$. Repeats the procedure for all rows in $\Delta'_1 := x_1\Delta_1$ as follows :

$\Delta'_2 = x_2\Delta_2$ with $\Delta'_3 = x_3\Delta_3, \Delta'_4 = x_4\Delta_4, \ldots, \Delta'_n = x_n\Delta_n$

$\Delta'_3 = x_3\Delta_3$ with $\Delta'_4 = x_4\Delta_4, \Delta'_5 = x_5\Delta_5, \ldots, \Delta'_n = x_n\Delta_n$

$\ldots$

$\Delta'_{n-1} = x_{n-1}\Delta_{n-1}$ with $\Delta'_n = x_n\Delta_n$.

To get the result of diagram graph of semigroup natural numbers, there are : $\frac{n^k}{2}\{k(n-1)-2\}+1$ free generators from our assumption.

Hence there are $n\left\{\frac{n^k}{2}\{k(n-1)-2-\right\} + \frac{n^{k+1}}{2}(n-1)+1$ free generators in $K_{k+1}(N)$. It is not hard to show that

$$n\left\{\frac{n^k}{2}\{k(n-1)-2-\right\} + \frac{n^{k+1}}{2}(n-1)+1 = \left\{\frac{n^{k+1}}{2}\left\{(k+1)(n-1)-2-\right\}\right\}+1$$

to complete the proof.

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