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On μ-Operators

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Abstract

In this paper we introduce a new class - [μ] - of operators acting on a complex Hilbert space H: If $T \in L(H)$ then $T \in [\,\mu\,]$ if $T^2 = -T^{*2}$. We investigate some basic properties of operators in[μ]. We study the relation between the class [μ] and some other well known classes of operators acting on H.

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1- Introduction

Let H be a complex Hilbert space and let $L(H)$ be the algebra of all bounded linear operators acting on H. If $T \in L(H)$ then T^* is its adjoint and $T = A + iB$ is its Cartesian decomposition. Many classes of operator in $L(H)$ are defined according to the relation between T and T^*, for example T is **normal** if and only if $TT^* = T^*T$; **2-normal** – [2] – if and only if $T^2T^* = T^*T^2$; **skew-normal** – [5] - if and only if

$T^2 = T^{*2}$; **quasinormal** –[1]- if and only if $TT^*T = T^*T^2$. In this paper we consider operators in $L(H)$ for which $T^2 = -T^{*2}$. The class of all such operators will be denoted by $[\,\mu\,]$. In section two we study some of the basic properties of operators in $[\,\mu\,]$. In section three we study the relation between the class $[\,\mu\,]$ and some other previously studied classes of operators in $L(H)$.

2. Preliminary notes

We start section two by a characterization of operators in $[\,\mu\,]$.

Proposition 2.1 If $T(= A + iB) \in L(H)$ then $T \in [\,\mu\,]$ if and only if $A^2 = B^2$.

Proof. By direct calculations we have

$$T^2 = (A^2 - B^2) + i(AB + BA) \quad -------- (i)$$

and

$$-T^{*2} = -(A^2 - B^2) + i(AB + BA) \quad -------- (ii)$$

Suppose first that $A^2 = B^2$ then clearly $T^2 = -T^{*2}$. Suppose now that $T^2 = -T^{*2}$ then it follows from (i) and (ii) above that $(A^2 - B^2) = -(A^2 - B^2)$ which implies that $A^2 = B^2$.

Proposition 2.2 If $T \in L(H)$ such that $T^2 = 0$ then $T \in [\,\mu\,]$.

Proof. Obvious .

Remark 2.1 It follows from proposition 2.2 that for each real number a each of the following operators acting on the two dimensional Hilbert space R^2 is in $[\,\mu\,]$:

$\begin{pmatrix} a & a \\ -a & -a \end{pmatrix}, \begin{pmatrix} -a & -a \\ a & a \end{pmatrix}, \begin{pmatrix} a & -a \\ a & -a \end{pmatrix}, \begin{pmatrix} -a & a \\ -a & a \end{pmatrix}$.

Proposition 2.3 If $S, T \in L(H)$ are unitarily equivalent and if $T \in [\,\mu\,]$ then so is S.

Proof. By assumption, there is a unitary operator $U \in L(H)$ such that $S = U^{-1}TU$ which implies that $S^* = U^*T^*(U^{-1})^* = U^*T^*(U^*)^{-1}$. Thus we have

$S^2 = U^{-1}TUU^{-1}TU = U^{-1}T^2U \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . (i)$.

and

$$-S^{*2} = -U^*T^*(U^*)^{-1}U^*T^*(U^*)^{-1} = -U^*T^{*2}(U^*)^{-1} \dots (ii)$$

Since U is unitary, $U^{-1} = U^*$ and using the fact that $T^2 = -T^{*2}$ we conclude that $U^{-1}T^2U = -U^*T^{*2}(U^*)^{-1}$. Thus $S^2 = -S^{*2}$, which implies that $S \in [\,\mu\,]$.

The following example shows that If $T \in [\,\mu\,]$ then it is not necessary that $T + kI \in [\,\mu\,]$ for all real numbers:

Example 2.1 Consider the operators $T = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ acting on R^2 then $T \in [\,\mu\,]$. Consider the operators $T + I = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = S$ (say) then by direct calculations one can show that $S^2 = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} \neq \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix} = -S^{*2}$. Thus $S \notin [\,\mu\,]$.

The following example shows that $[\,\mu\,]$ is not closed under addition or multiplication.

Example 2.2 Consider the two operators $T = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$, $F = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ acting on R^2 then $T, F \in [\,\mu\,]$. Consider $T + F = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = S$, (say) then $S^2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \neq \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} = -S^{*2}$. Thus $T + F \notin [\,\mu\,]$.

Now consider $TF = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 2I$,then $(TF)^2 = 4I \neq -4I = -(TF)^{*2}$. Thus $TF \notin [\,\mu\,]$.

Notice that $TF = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \neq \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} = -FT$

Proposition 2.4 If $T, F \in [\,\mu\,]$ such that $TF = -FT$ then $T + F \in [\,\mu\,]$.

Proof. Since $TF = -FT$ then $TF + FT = 0$ which implies that
$T^*F^* + F^*T^* = 0$. Now
$(T + F)^2 = T^2 + TF + FT + F^2 = T^2 + F^2$
$-(T + F)^{*2} = -(T^{*2} + T^*F^* + F^*T^* + F^{*2}) = -(T^{*2} + F^{*2})$
Since $-(T^{*2} + F^{*2}) = T^2 + F^2$, we have $(T + F)^2 = -(T + F)^{*2}$ which implies that $T + F \in [\,\mu\,]$.

Proposition 2.5 The direct sum and the tensor product of two operators in $[\,\mu\,]$ are in $[\,\mu\,]$.

Proof . Let $x = x_1 \oplus x_2$ be an element of $H \oplus H$ and let T and $\in [\,\mu\,]$. then

$$(T \oplus S)^2 x = (T \oplus S)^2 (x_1 \oplus x_2)$$

$$= (T^2 \oplus S^2)(x_1 \oplus x_2)$$
$$= T^2 x_1 \oplus S^2 x_2$$
$$= -T^{*2} x_1 \oplus -S^{*2} x_2$$
$$= -\left(T^{*2} \oplus S^{*2}\right)(x_1 \oplus x_2)$$

$$= -(T \oplus S)^{*2} x.$$

Thus $(T \oplus S)^2 = -(T \oplus S)^{*2}$. Hence $T \oplus S \in [\mu]$.

Also

$$(T \otimes S)^2 x = (T \otimes S)^2 (x_1 \otimes x_2)$$
$$= (T^2 \otimes S^2)(x_1 \otimes x_2)$$
$$= T^2 x_1 \otimes S^2 x_2$$
$$= -T^{*2} x_1 \otimes -S^{*2} x_2$$
$$= -\left(T^{*2} \otimes S^{*2}\right)(x_1 \otimes x_2)$$
$$= -(T \otimes S)^{*2} x.$$

Thus $(T \otimes S)^2 = -(T \otimes S)^{*2}$. Hence $T \otimes S \in [\mu]$.

The following example shows that $[\mu]$ is not convex :

Example 2.3 Consider the two operators $T = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$, $F = \begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix}$ acting on R^2 then $T, F \in [\mu]$. Consider $\frac{1}{2}T + \frac{1}{2}F = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = S$, (say) then $S^2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \neq \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} = -S^{*2}$. Thus $S \notin [\mu]$.

Proposition 2.6 The class $[\mu]$ is closed in the strong operator topology.

Proof. Let $\{\mu_n\}$ be a sequence of operators in $[\mu]$ that converges strongly to an operator S in $L(H)$ i.e. $\mu_n \overset{s}{\rightarrow} S$ then
$\|\mu_n x - S x\| \rightarrow 0$ as $n \rightarrow \infty$ for each $x \in H$. Thus
$\|\mu_n^* x - S^* x\| = \|(\mu_n - S)^* x\| \leq \|(\mu_n - S)^*\| \|x\| = \|\mu_n - S\| \|x\| \rightarrow 0$ as $n \rightarrow \infty$ .Thus $\mu_n^* \overset{s}{\rightarrow} S^*$. Since the product of operators is sequentially continuous in the strong operators topology – [1]- , $\mu_n^{*2} \overset{s}{\rightarrow} S^{*2}$,which implies that $-\mu_n^{*2} \overset{s}{\rightarrow} -S^{*2}$, and $\mu_n^2 \overset{s}{\rightarrow} S^2$. Since $\{\mu_n\}$ is a sequence of operators in $[\mu]$ then $-\mu_n^{*2} = \mu_n^2$ which implies that $\mu_n^2 \overset{s}{\rightarrow} -S^{*2}$. Since the limit is unique, $S^2 = -S^{*2}$. Thus $S \in [\mu]$ which implies that $[\mu]$ is closed in the strong operator topology.

3. Main results

In this section we study the relation between the class $[\,\mu\,]$ and some other classes of operators in $L(H)$. We start by showing that the class $[\,\mu\,]$ and some other classes of operators in $L(H)$ are independent.

Proposition 3.1 If $T \in L(H)$ is hermitian such that $T \in [\,\mu\,]$ then $T = 0$.
Proof . Since $\in [\,\mu\,]$, $T^2 = -T^{*2}$. Since T is hermitian , the last equation implies that $T^2 = 0$ which implies (Since T is hermitian) that $T = 0$.
Since there are nonzero hermitian operators (such as the identity operator I) and since there are nonzero operators in $[\,\mu\,]$ (for example any operator in remark 2.1) then we have:

Proposition 3.2 The class of all hermitian operators and the class $[\,\mu\,]$ are independent.

Proposition 3.3 The class of all normal operators and the class $[\,\mu\,]$ are independent.
Proof . A nonzero hermitian operator is a normal operator which is not in $[\,\mu\,]$.
The operator $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ is a nonnormal operator which is in $[\,\mu\,]$.

Proposition 3.4 If $T \in [\,\mu\,]$ such that T^2 is unitarily equivalent to T^* then T is normal.

Proof. Since T^2 is unitarily equivalent to T^*, there is a unitary operator U such that $T^* = UT^2U^*$ which implies that $T = UT^{*2}U^*$. Now it is easy to show that $TT^* = -UT^4U^* = T^*T$. Thus T is normal .

Proposition 3.5 The class of all skew-adjoint operators ($T = -T^*$) and the class $[\,\mu\,]$ are independent.

Proof. The operator $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ is a non-skew-adjoint operator which is in $[\,\mu\,]$.
The operator $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is a skew-adjoint operator which is not in $[\,\mu\,]$.

Proposition 3.6 If $T \in L(H)$ is skew-adjoint such that $T \in [\,\mu\,]$ then $T = 0$.

Proof. Let $T \in L(H)$ be skew-adjoint and let $T = A + iB$ be its Cartesian decomposition. Since T is skew-adjoint $T = -T^*$ which implies that $A = 0$. Thus $A^2 = 0$. Since $T \in [\,\mu\,]$ then , by ,Proposition 2.1, $B^2 = 0$ which implies (Since B is hermitian) that $B = 0$. Thus $T = 0$.

Proposition 3.7 The class of all isometric operators and the class $[\,\mu\,]$ are independent.

Proof. The identity operator I is an isometric operator and $I \notin [\,\mu\,]$.
The operator $T = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ is in $[\,\mu\,]$ but $T^*T = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \neq I$.
Proposition 3.8 If $T \in [\,\mu\,]$ is idempotent then $T = 0$.

Proof. Since $T \in [\,\mu\,], T^2 = -T^{*2}$. Since T is idempotent , $T^2 = T$ which implies that $-T^{*2} = -T^*$. Thus $T = -T^*$. Thus T is skew-adjoint. The result now follows from proposition 3.6

Corollary 3.1 If $T \in [\,\mu\,]$ is similar to an idempotent then $T = 0$

Proof. Since any operator similar to an idempotent is idempotent, T is idempotent. The result now follows immediately from proposition 3.8.

Proposition 3.9 The class of all idempotent operators and the class $[\,\mu\,]$ are independent.

Proof. We prove the result by the following two examples.

Example 3.1. Consider the operator $S = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ acting on R^2 then direct calculations shows that $S^2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = S$. Thus S is idempotent. However it can be shown that $S^2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \neq -S^{*2} = \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}$. Thus $S \notin [\,\mu\,]$.

Example 3.2. The operator $S = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$ acting on R^2 is in $[\,\mu\,]$ but $S^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq$ S **.** Thus S is not idempotent.

In [2] the author introduced the class of 2-normal operators in $L(H)$: $T = A + iB \in L(H)$ is called 2-normal if $A^2B = BA^2$ and $B^2A = AB^2$. Several characterizations of 2-normal operators were given in [2] such as : $T \in L(H)$ is 2-normal if and only if $T^2T^* = T^*T^2$; if and only if T^2 is normal. The class of all 2-normal operators is denoted by [2N].

Proposition 3.10 If $T \in L(H)$ is a μ −operator then $T \in$ [2N].

Proof . Let $= A + iB$. Since $T \in [\,\mu\,]$, $A^2 = B^2$. Multiplying the last equation on the left and then on the right by B we get
$A^2B = B^3 = BA^2$. Also Multiplying $A^2 = B^2$ on the left and then on the right by A we get $AB^2 = A^3 = B^2A$. Thus T is 2-normal.

Corollary 3.2 If $T,\ T + kI \in [\,\mu\,]$ for some nonzero complex number k then T is normal.

Proof. Since $T,\ T + kI \in [\,\mu\,]$ then – by proposition 3.10 - $T,\ T + kI$ are 2-normal operators. The result now follows from ([2] , proposition 3.3 , p. 193)

Remark 3.1 The converse of proposition 3.10 is not in general true. A nonzero hermitian operator in $L(H)$ is a 2-normal operator which is not in $[\,\mu\,]$.

Definition 3.1 If $T \in L(H)$ then T is called quasinormal if $TT^*T = T^*T^2$.

Proposition 3.11 If $T \in L(H)$ such that T is 2-normal andquasinormal then T is normal.

Proof. ([2] , proposition 2.3, p. 193).

Using proposition 3.11 we conclude two facts :

The first is that there are operators in $[\,\mu\,]$ which are not quasinormal since otherwise all operators in $[\,\mu\,]$ would be normal which is not true.
The second is that there are quasinormal operators which are not in $[\,\mu\,]$ since otherwise all quasinormal operators would be normal which is not true.

From the previous discussion we have:

Proposition 3.12 .The class $[\ \mu\ ]$ and the class of all quasinormal

operators are independent.

In [3] the author introduced the class of $\propto$ −operators : $T \in L(H)$ is called an $\propto$ − operator if $T^3 = T^*$. The class of all $\propto$ −operators is denoted by $(\propto)$.

In the following we give an example of a an operator which is in $[\ \mu\ ]$ but not in $(\propto)$:

Example 3.3 Consider the operators $T = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ acting on R^2 then $T \in [\ \mu\ ]$. Now it is easily shown that $T^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq T^* = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$. Thus $T \notin (\propto)$.
In the following we give an example of a an operator which is in
$(\propto)$ but not in $[\ \mu\ ]$:

Example 3.4 Consider the operators $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ acting on R^2 then $T^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = T^*$. Thus $T \in (\propto)$. However one can easily show that $T^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ while $-T^{*2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus $T \notin [\ \mu\ ]$.

Using the last two examples we conclude that

Proposition 3.13 The two classes $[\ \mu\ ]$ and $(\propto)$ are independent.

In [4] the author introduced the class of subprojection operators in $L(H)$: $T \in L(H)$ is called a subprojection if $T^2 = T^*$. The class of all subprojections is denoted by $S(H)$. In the following we give an example of an operator in $[\ \mu\ ]$ which is not in $S(H)$:

Example 3.5 Consider the operators $T = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ acting on R^2 then – by remark 2.1 - $T \in [\ \mu\ ]$. However $T^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = T^*$. Thus $T \notin S(H)$.

In the following we give an example of an operator in $S(H)$ which is not in $[\ \mu\ ]$:

Example 3.6 Consider the operators $T = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ acting on R^2then $T^2 =$
$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \neq -\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = -T^{*2}$.

Thus $T \notin [\, \mu \,]$. However one can easily show that $T^2 =$

$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = T^*$. Thus $T \in S(H)$.

We conclude from the last two examples:

Proposition 3.14 The two classes $[\, \mu \,]$ and $S(H)$ are independent.

Proposition 3.15 If $T \in [\, \mu \,] \cap S(H)$ then $T = 0$.

Proof. Since $T \in S(H)$, $T^2 = T^*$ which implies that $-T^{*2} = -T$.
Since $T \in [\, \mu \,]$, $T^2 = -T^{*2}$ which implies that $T = -T^*$. Thus if $T = A + iB$ then the last the last equation implies that $T + T^* = 0$ which implies that $A = 0$. Since $A^2 = B^2$, $B^2 = 0$ which implies (since B is hermitian) that $B = 0$. Thus $T = 0$.
In [6] Kutkut introduced a new class of operators which he called the class of parahyponormal operators : $T \in L(H)$ is called parahyponormal if $\|T\,x\|^2 \leq \|TT^*x\|$, for all x in H with $\|x\|$ =1,or equivalently, ([8] ,Theorem 1.1, p74), if and only if for every $\lambda > 0$. $\lambda^2 + T^*T \ \ (TT^*)^2 - 2\lambda$. The class of all parahyponormal operators is denoted by Phn(H). In the following we give an example of an operator which is in Phn(H) but not in $[\, \mu \,]$.

Example 3.7 The operator $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ acting on $R^2$is in Phn(H) ([6], p.83) but direct calculations shows that $T^2 =$
$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \neq \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix} = -T^{*2}$. Thus $T \notin [\, \mu \,]$.

In the following we give an example of an operator which is in $[\,\mu\,]$ but not in Phn(H).

Example 3.8 The operator $T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is not in Phn(H) ([6], p.81). However and by direct calculations one can show that $T^2 = 0$. Thus $T \in [\,\mu\,]$.
From the last two examples we conclude that

Proposition 3.16 The two classes $[\,\mu\,]$ and Phn(H) are independent.

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