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# **Double Stage Shrinkage Estimator of Two Parameters**

# **Generalized Exponential Distribution**

#### Alaa M. Hamad

Department of Mathematics Ibn Al-Haitham College of Education, Baghdad University, Baghdad, Iraq

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#### **Abstract**

This paper is concerned with double stage shrinkage estimator (DSSE) for lowering the mean squared error of classical estimator (MLE) for the shape parameter ( $\alpha$ ) of generalized Exponential (GE) distribution in a region (R) around available prior knowledge ( $\alpha_0$ ) about the actual value ( $\alpha$ ) as initial estimate in case when a scale parameter ( $\lambda$ ) is known as well as to reduce the cost of experimentations.

In situation where the experimentations are time consuming or very costly, a double stage procedure can be used to reduce the expected sample size needed to obtain the estimator.

This estimator is shown to have smaller mean squared error for certain choice of the shrinkage weight factor  $\psi(\cdot)$  and for acceptance mentioned region R.

Expressions for Bias, Mean square error (MSE), Expected sample size  $[E(n/\alpha,R)]$ , Expected sample size proportion  $[E(n/\alpha,R)/n]$ , probability for avoiding the second sample  $[p(\hat{\alpha} \in R)]$  and percentage of overall sample saved

$$\left[\frac{n_2}{n}p(\hat{\alpha} \in R)*100\right]$$
 for the proposed estimator are derived.

Numerical results and conclusions are established when the consider estimator (DSSE) are estimator of level of significance  $\Delta$ .

Comparisons with the classical estimator and with the last studies shown the usefulness of the proposed estimator

**Keywords:** Generalized Exponential Distribution; double stage shrinkage estimator; Maximum Likelihood Estimator; mean squared error

#### 1. Introduction

Gupta and Kundu (1999) proposed the generalized exponential (GE) distribution as an alternative to the well known Weibull or gamma distributions. It is observed that the proposed two-parameter GE distribution has several desirable properties and in many situations it may fit better than the Weibull or gamma distribution. Extensive work has been done since then to establish several properties of the generalized exponential distribution. The readers are referred to the recent review article by Gupta and Kundu (2007) for a current account of it.[1].Generalized exponential (GE) distribution has been proposed and studied quite extensively recently by Gupta and Kundu [2, 3, 4, 5, 6]. The readers may be referred to some of the related literature on  $\leftarrow GE \leftarrow distribution$ .[7].The two-parameters  $\leftarrow GE \leftarrow distribution$  has the following distribution function:

$$F(x; \alpha, \lambda) = [1 - e^{-(\lambda x)}]^{\alpha}$$
 for  $x > 0$ ,  $\alpha > 0$ ,  $\lambda > 0$  .....(1)

Thus, the probability density function (p.d.f.) of (GE) distribution is

$$f\left(x;\alpha,\lambda\right) = \begin{cases} \alpha\lambda x\,e^{-(\lambda x)}(1-e^{-(\lambda x)})^{\alpha-1} & \quad \text{for } x>0,\alpha,\lambda>0\\ 0 & \quad \text{o.w.} \end{cases} \tag{2}$$

Where,  $\alpha$  and  $\lambda$  are the shape and scale parameters respectively.

In this paper we introduce the problem of estimating of the shape parameter  $(\alpha)$  of GE distribution with known scale parameter  $(\lambda)$  when some prior information  $(\alpha_0)$  regarding the actual value  $(\alpha)$  available due past experiences such a prior estimate may arise for any one of an umber of reasons [8], e.g., we are estimating  $\alpha$  and;

- **i.** We believe  $\alpha_0$  is close to true value of  $\alpha$ , or
- ii. We fear that  $\alpha_0$  may be near the true value of  $\alpha$ , i.e.; something bad happens if  $\alpha = \alpha_0$  and we do not know about it.

In such a situation it is natural to start with an estimator  $\hat{\alpha}$  (e.g. MLE) of  $\alpha$  and modify it by moving it closer to  $\alpha_0$ , so that the resulting estimator, though perhaps biased, has smaller mean square error than that of  $\hat{\alpha}$  in some interval around  $\alpha_0$ . This method of constructing an estimator of  $\alpha$  that incorporates the prior value  $\alpha_0$  leads to what is known as a shrinkage estimator.

It is an important aspect of estimation that one should be able to get an estimator quickly using minimum experimentation. This also economizes cost of experimentation. To achieve this, double stage shrinkage estimator were introduced.

A double stage shrinkage estimator procedure is defined as follows:

Let  $x_{1i}$ ;  $i = 1, 2, ..., n_1$  be a random sample of  $n_1$  from GE distribution and  $\hat{\alpha}_1$  be a "good" estimator of  $\alpha$  based on these  $n_1$  observation. Construct a preliminary test region R in the parameter space based on  $\alpha_0$  and an appropriate criterion.

If  $\hat{\alpha}_1 \in R$  shrink  $\hat{\alpha}_1$  towards  $\alpha_0$  by shrinkage weight factor  $\psi(\hat{\alpha}) = e^{\frac{-10}{n1}}$  and use the shrinkage estimator  $\psi(\hat{\alpha}_1)\hat{\alpha}_1 + (1 - \psi(\hat{\alpha}_1))\alpha_0$ , for estimate  $\alpha$ .

If  $\hat{\alpha}_1 \notin R$ , obtain  $x_{2i}$ ;  $i=1, 2, ..., n_2$ , an additional sample of size  $n_2$  and use a pooled estimator  $\hat{\alpha}_p$  of  $\alpha$  based on combined sample of size  $n=n_1+n_2$ , i.e.;  $\hat{\alpha}_p = \frac{n_1\hat{\alpha}_1 + n_2\hat{\alpha}_2}{n}$ .

Thus, the double stage shrinkage estimator (DSSE) of  $\boldsymbol{\alpha}$  will be:

$$\tilde{\alpha} = \begin{cases} \psi_1(\hat{\alpha}_1)\hat{\alpha}_1 + (1 - \psi_1(\hat{\alpha}_1))\alpha_0 & \text{,if } \hat{\alpha}_1 \in \mathbb{R} \\ \hat{\alpha}_p & \text{,if } \hat{\alpha}_1 \notin \mathbb{R} \end{cases}$$
(3)

The motivation of this study was provided by the work of [9], [10] and [11].

The aim of this paper is to employ the double stage shrinkage estimator (DSSE)  $\tilde{\alpha}$  defined by (3) for estimate the shape parameter ( $\alpha$ ) of two parameters generalized Exponential (GE) distribution when the scale parameter ( $\lambda$ ) is known.

The expression of Bias, Mean squared error (MSE), Relative Efficiency [R.Eff(·)], Expected sample size, Expected sample size proportion, probability for avoiding the second sample and percentage of overall sample saved are derived and obtained for the estimator  $\tilde{\alpha}$ .

Numerical results and conclusions due mentioned expressions including some constants are performed and displayed in annexed tables.

Comparisons between the proposed estimator with the classical estimator  $(\hat{\alpha})$  and with some of the last studies are demonstrated.

## 2. Unbiased - Maximum Likelihood Estimator of $\alpha$

In this section, we consider the maximum likelihood estimator (MLE) of GE distribution with shape and scale parameter  $\alpha$  and  $\lambda$  respectively i.e. GE  $(\alpha, \lambda)$ .

Assume  $x_{11}, x_{12}, ..., x_{1_{n_1}}$  be a random sample of size  $n_1$  from  $GE(\alpha, \lambda)$  then the log-likelihood function  $L(\alpha, \lambda)$  can be written as:

$$L(\alpha,\lambda) = n_1 \ln \alpha + n_1 \ln \lambda + (\alpha - 1) \sum_{i=1}^{n_1} \ln (1 - e^{-(\lambda x_{1i})}) - \lambda \sum_{i=1}^{n_1} x_{1i} \dots (4)$$

In this paper we take  $\lambda = 1$  ( $\lambda$  is known).

So, 
$$\frac{\partial L}{\partial \alpha} = \frac{n_1}{\alpha} + \sum_{i=1}^{n_1} \ln(1 - e^{-x_{1i}}) = 0$$
 ....(5)

Then, the MLE of  $\alpha$ , say  $\hat{\alpha}_1$  is

$$\hat{\alpha}_{1} = -\frac{n_{1}}{\sum_{i=1}^{n_{1}} \ln(1 - e^{-x_{1i}})}...(6)$$

Note that, if  $x_{1i} \in {}^{iid}$  GE  $(\alpha, 1)$ , then  $-\alpha \sum_{i=1}^{n_1} \ln(1 - e^{-x_{1i}}) \sim G(n_1, 1)$ , see [4].

i.e.; 
$$E(\hat{\alpha}_1) = \frac{n_1}{n_1 - 1} \alpha$$
 and  $var(\hat{\alpha}_1) = \frac{n_1^2 \alpha^2}{(n_1 - 1)^2 (n_1 - 2)}$ .

Using (6), an unbiased estimator  $\hat{\alpha}_{_1}$  of  $\alpha$  can be easily obtained as:

$$\hat{\alpha}_{1} = \frac{n_{1} - 1}{n_{1}} \hat{\alpha}_{1} = -\frac{n_{1} - 1}{\sum_{i=1}^{n_{1}} \ln(1 - e^{-x_{1i}})}$$
 (7)

$$E(\hat{\alpha}_1) = \alpha \text{ and } var(\hat{\alpha}_1) = MSE(\hat{\alpha}_1) = \frac{\alpha^2}{n_1 - 2}$$
 (8)

## 3. Double Stage Shrinkage Estimator (DSSE) $\tilde{\alpha}$

In this section, we consider the (DSSE)  $\tilde{\alpha}$  which is defined in (3) using  $\hat{\alpha}_1$  defined by (7), when  $\psi(\hat{\alpha}_1) = k$  is a constant weight factor ( $k = e^{\frac{-10}{n1}}$ ) for estimate the shape parameter  $\alpha$  of GE distribution when  $\lambda = 1$ .

$$\tilde{\alpha} = \begin{cases} k\hat{\alpha}_1 + (1-k)\alpha_0 & \text{,if } \hat{\alpha}_1 \in R \\ \hat{\alpha}_p = \frac{n_1\hat{\alpha}_1 + n_2\hat{\alpha}_2}{n} & \text{,if } \hat{\alpha}_1 \notin R \end{cases}$$
 (9)

Where R is a pretest region for testing the hypothesis  $H_0$ :  $\alpha = \alpha_0$  vs  $H_A$ :  $\alpha \neq \alpha_0$  with level of significance ( $\Delta$ ) using test statistic function  $T(\hat{\alpha}_1/\alpha_0) = \frac{2(n_1-1)\alpha_0}{\hat{\alpha}_0}$ 

i.e.; 
$$R = \left[ \frac{2(n_1 - 1)\alpha_0}{b}, \frac{2(n_1 - 1)\alpha_0}{a} \right]$$
 ....(10)

The expression for Bias of DSSE ( $\tilde{\alpha}$ ) is defined as below

$$\begin{split} Bias(\tilde{\alpha}/\alpha,R) &= E(\tilde{\alpha}-\alpha) \\ &= \int\limits_{\hat{\alpha}_2=0}^{\infty} \int\limits_{\hat{\alpha}_1 \in R} \left[ k(\hat{\alpha}_1 - \alpha_0) + (\alpha_0 - \alpha) \right] f(\hat{\alpha}_1) f(\hat{\alpha}_2) d\hat{\alpha}_1 d\hat{\alpha}_2 + \\ &\int\limits_{\hat{\alpha}_2=0}^{\infty} \int\limits_{\hat{\alpha}_1 \notin \overline{R}} \left[ \hat{\alpha}_p - \alpha \right] f(\hat{\alpha}_1) f(\hat{\alpha}_2) d\hat{\alpha}_1 d\hat{\alpha}_2 \end{split}$$

Where R is the complement region of R in real space and

$$f(\hat{\alpha}_i) = \begin{cases} \left[\frac{(n_i - 1)\alpha}{\hat{\alpha}_i}\right]^{n_i + 1} e^{-\frac{(n_i - 1)\alpha}{\hat{\alpha}_i}} \\ \frac{\Gamma(n_i) \cdot (n_i - 1)\alpha}{0} & \text{for } \hat{\alpha}_i > 0, \alpha > 0 \cdot \dots (12) \end{cases}$$

We conclude,

Bias(
$$\tilde{\alpha}/\alpha$$
, R) =  $\alpha \left\{ (K+1)(\zeta-1)J_0(a^*,b^*) + \frac{1}{1+u}[(n_1-1)J_1(a^*,b^*) - J_0(a^*,b^*)] \right\} \dots (13)$ 

Where 
$$J_{\ell}(a^*, b^*) = \int_{a^*}^{b^*} y^{-\ell} \frac{y^{n_1-1} e^{-y}}{\Gamma(n_1)} dy; \ell = 0, 1, 2, \dots, (14)$$

Also 
$$y = \frac{(n_1 - 1)\alpha}{\hat{\alpha}_1}$$
 ....(15)

The Bias ratio [B (·)] of DSSE (
$$\tilde{\alpha}$$
) is defined as: 
$$B(\tilde{\alpha}) = \frac{Bias(\tilde{\alpha}/\alpha, R)}{\alpha}....(16)$$

The expression of Mean squared error [MSE (·)] of  $\tilde{\alpha}$  derived as below:- $MSE(\tilde{\alpha}/\alpha, R) = E(\tilde{\alpha} - \alpha)^2$ 

$$\begin{split} &=\int\limits_{\hat{\alpha}_{2}=0}^{\infty}\int\limits_{\hat{\alpha}_{1}\in R}\left[k(\hat{\alpha}_{1}-\alpha)+(\alpha_{0}-\alpha)\right]^{2}f(\hat{\alpha}_{1})f(\hat{\alpha}_{2})d\hat{\alpha}_{1}d\hat{\alpha}_{2} + \\ &\int\limits_{\hat{\alpha}_{2}=0}^{\infty}\int\limits_{\hat{\alpha}_{1}\notin \overline{R}}\left[\hat{\alpha}_{p}-\alpha\right]^{2}f(\hat{\alpha}_{1})f(\hat{\alpha}_{2})d\hat{\alpha}_{1}d\hat{\alpha}_{2} \end{split}$$

And by simple computations, one can get:

$$\begin{split} & \begin{cases} k^2 \bigg[ (n_1 - 1)^2 J_2(a^*, b^*) - 2\zeta(n_1 - 1) J_1(a^*, b^*) + \zeta^2 J_0(a^*, b^*) \bigg] \\ + 2k(\zeta - 1) \bigg[ (n_1 - 1) J_1(a^*, b^*) - \zeta J_0(a^*, b^*) \bigg] + \\ & (\zeta - 1)^2 J_0(a^*, b^*) + \\ & \bigg[ \bigg( \frac{1}{1 + u} \bigg)^2 \bigg[ \bigg( \frac{1}{n_1 - 2} \bigg) - [(n_1 - 1)^2 J_2(a^*, b^*) + 2(n_1 - 1) J_1(a^*, b^*) - J_0(a^*, b^*) \bigg] + \\ & \bigg[ \bigg( \frac{u}{1 + u} \bigg)^2 \bigg( \frac{1}{n_1 u - 2} \bigg) \bigg[ 1 - J_0(a^*, b^*) \bigg] \end{split} \end{split}$$

Now, the Efficiency of  $\tilde{\alpha}$  relative to  $\hat{\alpha}$  denote by R.Eff ( $\tilde{\alpha}/\alpha$ , R) is defined by:

$$R.Eff(\tilde{\alpha}/\alpha, R) = \frac{MSE(\hat{\alpha})}{MSE(\tilde{\alpha}/\alpha, R) \cdot [E(n/\alpha, R/n]]}...(18)$$

Where  $E(n/\alpha,R)$  is the Expected sample size, which is defined as:

$$E(n/\alpha, R) = n \left[ 1 - \frac{u}{1+u} J_0(a^*, b^*) \right]$$
 (19)

See for example [10] and [11].

As well as the Expected sample size proportion E  $(n/\alpha, R)/n$  equal to  $1 - \frac{u}{1+u} J_0(a^*, b^*) \dots (20)$ 

Also, we have to define the percentage of the overall sample saved (p.o.s.s.) of  $\tilde{\alpha}$  as:

p.o.s.s. = 
$$\frac{n_2}{n} J_0(a^*, b^*) * 100$$
 .....(21)

And, finally,  $P(\hat{\alpha} \in R)$  represent the probability of a voiding the second sample (stage).

### 4. Conclusions and Numerical Results

The computations of Relative Efficiency [R.Eff( $\cdot$ )] and Bias Ratio [B( $\cdot$ )], Expected sample size  $[E(n/\alpha,R)]$ , Expected sample size proportion  $[E(n/\alpha,R)/n]$ , Percentage of the overall sample saved (p.o.s.s.) and probability of a voiding the second sample  $[P(\hat{\alpha} \in R)]$  were used for the estimator  $\tilde{\alpha}$ . These computations were performed for  $n_1 = 4$ , 8, 16, 20 u (=  $n_2/n_1$ ) = 2, 6, 10, 12,  $\zeta$  (=  $\alpha_0/\alpha$ ) = 0.25(0.25)2,  $\Delta = 0.01$ , 0.05, and  $k = \psi(\hat{\alpha}) = e^{\frac{-10}{n1}}$ 

Some of these computations are given in the tables (1)-(4).

The observation mentioned in the tables lead to the following results:

- i. The Relative Efficiency [R.Eff(·)] of  $\tilde{\alpha}$  are adversely proportional with small value of  $\Delta$  especially when  $\zeta = 1$ , i.e.  $\Delta = 0.01$  yield highest efficiency
- ii. The Relative Efficiency [R.Eff(·)] of  $\tilde{\alpha}$  has maximum value when  $\alpha = \alpha_0(\zeta = 1)$ , for each  $n_1$ ,  $\Delta$ , and decreasing otherwise ( $\zeta \neq 1$ ). This feature shown the important usefulness of prior knowledge which given higher effects of proposed estimator as well as the important role of shrinkage technique and its philosophy.
- iii. Bias ratio [B  $(\cdot)$ ] of  $\tilde{\alpha}$  are reasonably small when  $\alpha = \alpha_0$  for each  $n_1$ ,  $\Delta$ , and increases otherwise. This property shown that the proposed estimator  $\tilde{\alpha}$  is very closely to unbiasedness property especially when  $\alpha = \alpha_0$ .
- iv. The Effictive interval of  $\tilde{\alpha}$  [the value of  $\tilde{\alpha}$  which makes R.Eff(·) of  $\tilde{\alpha}$  greater than one] is [0.5, 1.5].
- **v.** Bias ratio [B (·)] of  $\tilde{\alpha}$  are increases with small value of u.
- vi. R.Eff( $\tilde{\alpha}$ ) is decreasing function with increasing of the first sample size  $n_1$ , for each  $\Delta$  and  $\zeta$ .
- vii. The expected values of sample size of  $\tilde{\alpha}$  are close to  $n_1$ , especially when  $\zeta \ge 1$  and start far-away otherwise.
- **viii.** Percentage of the overall sample saved  $\left[\frac{n_2}{n}J_0(a^*,b^*)*100\right]$  is increasing value with increasing value of u ( $u = n_2 / n_1$ ) and  $\zeta$ .
  - ix. R.Eff( $\tilde{\alpha}$ ) is an increasing function with respect to u. This property shown the effective of proposed estimator using small  $n_1$  relative to  $n_2$  (or large  $n_2$ ) which given higher efficiency and reduce the observation cost.
  - **x.** The considered estimator  $\tilde{\alpha}$  is better than the classical estimator especially when  $\alpha \approx \alpha_0$ , this will given the effective of  $\tilde{\alpha}$  relative to  $\hat{\alpha}$  and also given an important weight of prior knowledge, and the augmentation of efficiency may be reach to tens times.
  - **xi.**The considered estimator  $\tilde{\alpha}$  is more efficient than the estimators introduced by [8], in the sense of higher efficiency.

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Table (1) Shown Bias ratio [B (·)] and R.E.ff of  $\tilde{\alpha}$  w.r.t  $\Delta$ ,  $n_1$  and  $\zeta$  when u=2

		•	ζ					
Δ	n <sub>1</sub>	R.Eff. Bias	0.25	0.75	1	1.5	1.75	2
	4	$R.Eff(\cdot)$	0.525	2.386	6.722	1.257	0.601	0.345
		$B(\cdot)$	-0.215	-0.295	-0.035	0.517	0.793	1.064
	8	$R.Eff(\cdot)$	0.9	1.21	4.882	0.978	0.456	0.259
0.01	0	$B(\cdot)$	-6.319*10^-4	-0.249	-0.037	0.616	0.947	1.271
0.01	16	$R.Eff(\cdot)$	0.956	0.761	1.806	1.01	0.478	0.277
		$B(\cdot)$	-3.902*10^-1	-0.073	-0.034	0.716	1.127	1.519
	20	$R.Eff(\cdot)$	0.965	0.811	1.288	1.076	0.509	0.297
		$B(\cdot)$	0	-0.029	-0.028	0.735	1.177	1.59
	4	$R.Eff(\cdot)$	0.723	1.591	3.851	1.147	0.569	0.329
		$B(\cdot)$	-0.027	-0.255	-0.065	0.468	0.744	1.01
	8	$R.Eff(\cdot)$	0.904	0.94	2.616	0.977	0.451	0.252
0.05	٥	$B(\cdot)$	-3.918*10^-6	-0.134	-0.05	0.552	0.894	1.215
0.05	16	$R.Eff(\cdot)$	0.956	0.878	1.157	1.168	0.503	0.273
	10	$B(\cdot)$	0	-0.015	-0.025	0.612	1.056	1.455
	20	$R.Eff(\cdot)$	0.965	0.93	0.975	1.324	0.549	0.295
	20	$B(\cdot)$	0	-3.98*10^-3	0.016	0.611	1.097	1.523

Table~(2) Shown Bias ratio [B (·)] and R.E.ff of  $\tilde{\alpha}~$  w.r.t  $\Delta,$   $n_1$  and  $\zeta$  when u=6

			ζ						
Δ	n <sub>1</sub>	R.Eff. Bias	0.25	0.75	1	1.5	1.75	2	
	4	R.Eff(·)	0.312	2.514	22.294	1.18	0.54	0.304	
		$B(\cdot)$	-0.193	-0.264	-0.015	0.527	0.799	1.068	
	8	$R.Eff(\cdot)$	0.943	0.832	7.295	0.889	0.42	0.239	
0.01	0	$B(\cdot)$	-5.631*10^-4	-0.221	-0.016	0.623	0.951	1.272	
0.01	16	$R.Eff(\cdot)$	0.98	0.501	1.462	0.872	0.444	0.261	
		$B(\cdot)$	-3.501*10^-1	-0.064	-0.015	0.723	1.13	1.519	
	20	$R.Eff(\cdot)$	0.985	0.606	0.93	0.902	0.471	0.281	
		$B(\cdot)$	0	-0.025	-0.012	0.743	1.18	1.59	
	4	$R.Eff(\cdot)$	0.692	1.417	8.917	0.984	0.464	0.259	
		$B(\cdot)$	-0.024	-0.213	-0.028	0.49	0.76	1.02	
	8	$R.Eff(\cdot)$	0.956	0.652	3.074	0.77	0.369	0.206	
0.05	٥	$B(\cdot)$	-3.453*10^-6	-0.112	-0.021	0.569	0.904	1.219	
0.03	16	$R.Eff(\cdot)$	0.98	0.748	0.889	0.858	0.405	0.228	
	10	$B(\cdot)$	0	-0.013	-0.011	0.626	1.064	1.457	
	20	$R.Eff(\cdot)$	0.985	0.88	0.731	0.96	0.437	0.248	
	20	$B(\cdot)$	0	-3.365*10^-3	-6.765*10^-3	0.625	1.104	1.525	

Table (3) Shown Bias ratio [B (·)] and R.E.ff of  $\tilde{\alpha}$  w.r.t  $\Delta,$   $n_1$  and  $\zeta$  when u=10

					ζ			
Δ	n <sub>1</sub>	R.Eff. Bias	0.25	0.75	1	1.5	1.75	2
	4	$\begin{array}{c} \text{R.Eff}(\cdot) \\ \text{B}(\cdot) \end{array}$	0.216 -0.186	2.129 -0.256	36.212 -9.59*10^-3	1.108 0.529	0.51 0.801	0.285 1.069
0.01	8	$\begin{array}{c} \text{R.Eff}(\cdot) \\ \text{B}(\cdot) \end{array}$	0.95 -5.44*10^-4	0.605 -0.213	7.064 -0.01	0.815 0.624	0.398 0.952	0.227 1.273
	16	$\begin{array}{c} R.Eff(\cdot) \\ B(\cdot) \end{array}$	0.987 -3.392*10^-1	0.37 -0.061	1.143 -9.307*10^-3	0.754 0.724	0.417 1.131	0.25 1.52
	20	$\begin{array}{c} R.Eff(\cdot) \\ B(\cdot) \end{array}$	0.99 0	0.482 -0.024	0.705 -7.706*10^-3	0.759 0.745	0.439 1.18	0.27 1.59
	4	$\begin{array}{c} R.Eff(\cdot) \\ B(\cdot) \end{array}$	0.616 -0.023	1.106 -0.201	12.111 -0.018	0.83 0.496	0.396 0.765	0.219 1.023
0.05	8	$R.Eff(\cdot)$ $B(\cdot)$	0.971 -3.327*10^-6	0.485 -0.107	2.856 -0.014	0.62 0.573	0.313 0.907	0.176 1.22
0.03	16	$R.Eff(\cdot)$ $B(\cdot)$	0.987 0	0.645 -0.012	0.705 -6.792*10^-3	0.655 0.63	0.337 1.066	0.197 1.458
	20	$R.Eff(\cdot)$ $B(\cdot)$	0.99 0	0.826 -3.19*10^-3	0.58 -4.305*10^-3	0.722 0.629	0.36 1.106	0.214 1.525

Table~(4) Shown Bias ratio [B (·)] and R.E.ff of  $\tilde{\alpha}~w.r.t~\Delta,\,n_1$  and  $\zeta$  when u=12

			ζ						
Δ	n <sub>1</sub>	R.Eff. Bias	0.25	0.75	1	1.5	1.75	2	
	4	$R.Eff(\cdot)$ $B(\cdot)$	0.187 -0.185	1.956 -0.254	41.524 -8.117*10^-3	1.078 0.53	0.498 0.801	0.278 1.07	
0.01	8	$\begin{array}{c} \text{R.Eff}(\cdot) \\ \text{B}(\cdot) \end{array}$	0.95 -5.393*10^-4	0.531 -0.211	6.731 -8.48*10^-3	0.783 0.625	0.388 0.952	0.222 1.273	
0.01	16	$\begin{array}{c} \text{R.Eff}(\cdot) \\ \text{B}(\cdot) \end{array}$	0.989 -3.362*10^-1	0.327 -0.06	1.025 -7.875*10^-3	0.706 0.725	0.405 1.131	0.245 1.52	
	20	$R.Eff(\cdot)$ $B(\cdot)$	0.992 0	0.437 -0.024	0.627 -6.52*10^-3	0.703 0.745	0.425 1.181	0.264 1.59	
	4	$R.Eff(\cdot)$ $B(\cdot)$	0.58 -0.022	0.988 -0.198	13.181 -0.015	0.769 0.498	0.37 0.766	0.204 1.024	
0.05	8	$R.Eff(\cdot)$ $B(\cdot)$	0.975 -3.292*10^-6	0.429 -0.105	2.712 -0.011	0.565 0.574	0.292 0.907	0.165 1.22	
0.05	16	$\begin{array}{c} R.Eff(\cdot) \\ B(\cdot) \end{array}$	0.989 0	0.603 -0.012	0.638 -5.747*10^-3	0.584 0.631	0.311 1.067	0.184 1.458	
	20	$\begin{array}{c} \text{R.Eff}(\cdot) \\ \text{B}(\cdot) \end{array}$	0.992 0	0.801 -3.151*10^-3	0.525 -3.643*10^-3	0.641 0.63	0.331 1.107	0.2 1.526	

 $Table\ (5)$  Shown Probability of a Voiding Second Sample w.r.t  $\Delta,\,u,\,n_1$  and  $\zeta$ 

					ζ			
u	$\mathbf{n_1}$	Δ	0.25	0.75	1	1.5	1.75	2
2	4	0.01	0.216	0.893	0.952	0.986	0.991	0.99
	4	0.05	0.026	0.668	0.823	0.937	0.952	0.95
6	8	0.01	53×10 <sup>-4</sup>	0.62	0.851	0.976	0.988	0.99
0	0	0.05	$322 \times 10^{-6}$	0.3	0.612	0.903	0.944	0.95
10	16	0.01	$278 \times 10^{-12}$	0.147	0.544	0.948	0.983	0.99
10	10	0.05	0	0.029	0.264	0.83	0.929	0.95
12	20	0.01	0	0.055	0.409	0.931	0.981	0.99
12	20	0.05	0	$722 \times 10^{-3}$	0.159	0.791	0.921	0.95

Table~(6) Shown Expected Sample Size of ~  $\tilde{\alpha}$  w.r.t  $\Delta$ , u, and  $\zeta$  when  $n_1=4$ 

		ζ							
u	Δ	0.25	0.75	1	1.5	1.75	2		
2	0.01	10.271	4.859	4.381	4.108	4.075	4.080		
	0.05	11.793	6.655	5.414	4.503	4.384	4.40		
-	0.01	22.812	6.578	5.143	4.325	4.226	4.240		
6	0.05	27.38	11.965	8.242	5.508	5.153	5.20		

 $Table\ (7)$  Shown Expected Sample Size Proportion w.r.t \ \Delta, u, n\_1 and  $\zeta$ 

						ζ		
u	$\mathbf{n}_1$	Δ	0.25	0.75	1	1.5	1.75	2
2	4	0.01	0.856	0.405	0.365	0.342	0.340	0.340
		0.05	0.983	0.555	0.451	0.375	0.365	0.367
6	8	0.01	1	0.468	0.27	0.164	0.153	0.151
0		0.05	1	0.743	0.475	0.226	0.190	0.186
10	16	0.01	1	0.866	0.496	0.138	0.106	0.1
10		0.05	1	0.974	0.760	0.246	0.155	0.136
12	20	0.01	1	0.949	0.623	0.14	0.095	0.086
12		0.05	1	0.993	0.853	0.27	0.150	0.123

 $Table\ (8)$  Shown Percentage of Overall Sample Saved w.r.t  $\Delta,$  u,  $n_1$  and  $\zeta$ 

					ζ			
u	$\mathbf{n}_1$	Δ	0.25	0.75	1	1.5	1.75	2
2	4	0.01	14.410	59.505	63.492	65.726	66.039	66.000
	4	0.05	1.723	44.542	54.883	62.4604	63.463	63.333
	. 0	0.01	0.045	53.157	72.976	83.657	84.715	84.857
6	8	0.05	276×10 <sup>-4</sup>	25.698	52.493	77.396	80.954	81.428
10	16	0.01	253×10 <sup>-10</sup>	13.408	50.397	86.1818	89.404	89.999
10	10	0.05	135×10 <sup>-1 4</sup>	2.644	24.032	75.447	84.456	86.363
12	20	0.01	$292 \times 10^{-15}$	5.3097	37.734	85.938	90.537	91.385
12	20	0.05	218×10 <sup>-20</sup>	0.667	14.655	73.0148	85.028	87.692

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