

# Chaos of Product Map on $G$ -Spaces

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**Abstract.** We obtain sufficient conditions under which product of two maps, in which one is Devaney's  $G_1$ -chaotic and other is Devaney's  $G_2$ -chaotic, is Devaney's  $G_1 \times G_2$ -chaotic.

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## 1. Introduction

Over the past several years, there has been tremendous interest in studying the behavior of chaotic systems. They are characterized by sensitive dependence on initial conditions, similarity to random behavior, and continuous broad-band power spectrum. Chaos has potential applications in several functional blocks of a digital communication system: compression, encryption and modulation. The possibility for self-synchronization of chaotic oscillations has sparked an avalanche of works on application of chaos in cryptography.

Probably the most widely accepted definition of chaos is the one given by Devaney which we call as Devaney's chaos [14]. In [16], Tian and Chen have defined and studied Devaney's chaos for a sequence of maps in iterative and successive ways on a metric space and have given several interesting examples to illustrate such chaotic systems. In [17], authors study uniform convergence, mixing and chaos. In [18], Devaney's chaos of uniform limit functions is studied. In [1, 2, 15], topological transitivity of uniform limit functions is studied. In [8], we have studied topological transitivity of uniform limit functions on

metric  $G$ -spaces. In [11], we have defined the notions of positively and negatively  $G$ -asymptotic points for a homeomorphism on a metric  $G$ -space. Also we have shown that the problem of studying  $G$ -expansive homeomorphisms on a bounded subset of a normed linear  $G$ -space is equivalent to the problem of studying linear  $G$ -expansive homeomorphisms on a bounded subset of another normed linear  $G$ -space. In [12], we have defined  $G$ -chaos for a sequence of maps in iterative and successive ways on a metric  $G$ -space and have given some interesting examples to illustrate such  $G$ -chaotic systems. In [13], author has obtained sufficient conditions under which product of two chaotic maps is chaotic in the sense of Devaney.

In Section 2, we give preliminaries required for Section 3. In Section 3, we obtain sufficient conditions under which product of two maps, in which one is Devaney's  $G_1$ -chaotic and other is Devaney's  $G_2$ -chaotic, is Devaney's  $G_1 \times G_2$ -chaotic.

## 2. Preliminaries

Let  $\mathbb{N}$  denote the set of positive integers,  $(X, d)$  denote metric space  $X$  with metric  $d$  and  $f : X \rightarrow X$  be a continuous map. Then  $f$  is said to have dense set of periodic points if the set of all periodic points of  $f$  is dense in  $X$ . If for any two non-empty open subsets  $U$  and  $V$  of  $X$ , there exists  $n \in \mathbb{N}$  such that  $f^n(U) \cap V \neq \emptyset$  then  $f$  is called topologically transitive on  $X$ . If there exists  $\delta > 0$  such that for every  $x \in X$  and any neighborhood  $U$  of  $x$ , there exists  $y \in U$  and an integer  $n \geq 0$  such that  $d(f^n(x), f^n(y)) > \delta$  then  $f$  is said to be sensitive dependence on initial conditions.

The map  $f$  is said to be chaotic on  $X$  in the sense of Devaney if (i)  $f$  is transitive on  $X$ ; (ii) the set of periodic points of  $f$  is dense in  $X$ ; and (iii)  $f$  is sensitive dependence on initial conditions. In [3], Banks et al have shown that if  $X$  is compact then (iii) is implied by other two conditions. Let  $(X, d)$  be a metric space and  $f : X \rightarrow X$ . If for every pair of non-empty open sets  $U, V$  of  $X$ , there exists an  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$ ,  $f^n(U) \cap V \neq \emptyset$  then  $f$  is called topologically mixing. The product of two topologically mixing maps is topologically mixing [13]. In [13], author has proved the following result:

**Theorem 2.1.** *Let  $(X, d_1)$  be a metric space,  $(Y, d_2)$  be a metric space,  $f : X \rightarrow X$ ,  $h : Y \rightarrow Y$  be chaotic and topologically mixing maps then  $f \times h : X \times Y \rightarrow X \times Y$  is chaotic.*

By a  $G$ -space we mean a triple  $(X, G, \theta)$ , where  $X$  is a Hausdorff space,  $G$  is a topological group and  $\theta : G \times X \rightarrow X$  is a continuous action of  $G$  on  $X$  [6]. Henceforth,  $\theta(g, x)$  will be denoted by  $gx$ . For  $A \subset X, g \in G$  we have  $gA = \{ga | a \in A\}$ . For  $x \in X$ , the set  $G(x) = \{gx | g \in G\}$ , is called the  $G$ -orbit of  $x$  in  $X$ . By trivial action of  $G$  on  $X$  we mean  $gx = x$  for all  $g \in G, x \in X$ . If  $X, Y$  are  $G$ -spaces, then a continuous map  $h : X \rightarrow Y$  is called *equivariant* if  $h(gx) = gh(x)$ , for each  $g$  in  $G$  and each  $x$  in  $X$ . We call  $h$  *pseudoequivariant* if  $h(G(x)) = G(h(x))$ , for each  $x$  in  $X$ . An equivariant

map is clearly pseudoequivariant but converse is not true [7]. In [7, 9, 10] several interesting results using pseudoequivariant maps have been obtained. By a *metric  $G$ -space*, we mean a metric space  $(X, d)$  on which a topological group  $G$  acts. If  $(X, d_1)$  is a metric  $G_1$ -space and  $(Y, d_2)$  is a metric  $G_2$ -space then  $X \times Y$  is a metric  $G_1 \times G_2$ -space under the metric  $d$  defined by  $d((x, y), (x', y')) = d_1(x, x') + d_2(y, y')$  and action of  $G_1 \times G_2$  on  $X \times Y$  defined by  $(g, k)(x, y) = (gx, ky)$ .

We recall the following notion defined in [8]:

**Definition 2.2.** Let  $(X, d)$  be a metric  $G$ -space and  $f : X \rightarrow X$  be a continuous map then  $f$  is called  $G$ -transitive if for every pair of non-empty open subsets  $U$  and  $V$  of  $X$ , there exists  $n \in \mathbb{N}$  and  $g \in G$  such that  $g.f^n(U) \cap V \neq \emptyset$ .

**Remark 2.3.** Under trivial action of  $G$  on  $X$ , notions of transitive and  $G$ -transitive coincide. Under non-trivial action of  $G$  on  $X$ , if  $f$  is transitive then it is  $G$ -transitive. But examples show that every  $G$ -transitive map need not be transitive [8].

**Definition 2.4.** Let  $(X, d)$  be a metric  $G$ -space and  $f : X \rightarrow X$  be a continuous map then  $x \in X$  is called  $G$ -periodic point of  $f$  if for some  $g \in G$  and some  $n \in \mathbb{N}$ ,  $f^n(x) = gx$ .

**Definition 2.5.** [12] Let  $(X, d)$  be a metric  $G$ -space and  $f : X \rightarrow X$  be continuous, then  $f$  is said to be  $G$ -sensitive dependence on initial conditions if there exists a  $\delta > 0$  such that for any  $x \in X$  and any neighborhood  $U$  of  $x$  in  $X$ , there exists a  $y \in U$  with  $G(x) \neq G(y)$  and a positive integer  $k$  such that  $d(f^k(gx), f^k(py)) > \delta$ , for all  $g, p$  in  $G$ .

**Remark 2.6.** Under trivial action of  $G$  on  $X$ , the above notion coincides with sensitive dependence on initial conditions [16].

**Definition 2.7.** [12] Let  $(X, d)$  be a metric  $G$ -space and  $f : X \rightarrow X$  be continuous. Then  $f$  is said to be  $G$ -chaotic in the sense of Devaney if following conditions are satisfied:

- (i)  $f$  is  $G$ -transitive on  $X$ .
- (ii) the set of  $G$ -periodic points of  $f$  is dense in  $X$ .
- (iii)  $f$  has  $G$ -sensitive dependence on initial condition.

### 3. Devaney's $G$ -Chaos of product map

We have the following result for product map:

**Theorem 3.1.** Let  $(X, d_1)$  be a metric  $G_1$ -space,  $(Y, d_2)$  be a metric  $G_2$ -space and  $f : X \rightarrow X$ ,  $h : Y \rightarrow Y$  be continuous maps. If  $f$  is  $G_1$ -sensitive dependence on initial conditions or  $h : Y \rightarrow Y$  is  $G_2$ -sensitive dependence on initial conditions then  $f \times h$  is  $G_1 \times G_2$ -sensitive dependence on initial conditions.

*Proof.* Suppose  $f$  is  $G_1$ -sensitive dependence on initial conditions. Let  $(x, y) \in X \times Y$  and  $W$  be any neighborhood of  $(x, y)$  in  $X \times Y$  then there exist an open set  $U$  of  $X$  containing  $x$  and an open set  $V$  of  $Y$  containing  $y$  such that  $U \times V \subseteq W$ . Since  $f$  is  $G_1$ -sensitive dependence on initial conditions therefore there exists an  $\varepsilon > 0$  such that for a certain  $x' \in U$  with  $G_1(x) \neq G_1(x')$  and integer  $n > 0$  satisfying  $d_1(f^n(gx), f^n(px')) > \varepsilon$ , for all  $g, p \in G_1$ . Now for any  $y' \in V$ ,  $(x', y') \in U \times V \subseteq W$ ,  $(G_1 \times G_2)(x, y) \neq (G_1 \times G_2)(x', y')$  and for all  $(g_1, g_2), (k_1, k_2) \in G_1 \times G_2$  we have,

$$\begin{aligned} d((f \times h)^n((g_1, g_2)(x, y)), (f \times h)^n(k_1, k_2)(x', y')) \\ &= d_1((f^n(g_1x), f^n(k_1x')) + d_2((h^n(g_2y), h^n(k_2y'))) \\ &\geq (d_1(f^n(g_1x), f^n(k_1x'))) \\ &> \varepsilon. \end{aligned}$$

Hence  $(f \times h)$  is  $G_1 \times G_2$ -sensitive dependence on initial conditions.  $\square$

**Theorem 3.2.** Let  $(X, d_1)$  be a metric  $G_1$ -space,  $(Y, d_2)$  be a metric  $G_2$ -space and  $f : X \rightarrow X$ ,  $h : Y \rightarrow Y$  be equivariant maps. If the set of  $G_1$ -periodic points of  $f$  is dense in  $X$  and the set of  $G_2$ -periodic points of  $h$  is dense in  $Y$  then the set of  $G_1 \times G_2$ -periodic points of  $f \times h$  is dense in  $X \times Y$ .

*Proof.* Let  $W$  be a non-empty open set of  $X \times Y$  then there exist a non-empty open set  $U$  of  $X$  and a non-empty open set  $V$  of  $Y$  such that  $U \times V \subseteq W$ . By hypothesis there exists  $x \in U$  such that  $f^n(x) = gx$  for some  $g \in G_1$  and some  $n \in N$ . Similarly there exists  $y \in V$  such that  $h^m(y) = py$  for some  $p \in G_2$  and some  $m \in N$ . Now for  $(x, y) \in U \times V \subseteq W$  and  $r = m.n$ , using equivariance of  $f$  and  $h$  we have

$$\begin{aligned} (f \times h)^r(x, y) &= (f \times h)^{m.n}(x, y) = (f^{m.n}(x), h^{m.n}(y)) \\ &= ((f^n)^m(x), (h^m)^n(y)) = (g^m x, p^n y) = (g^m, p^n)(x, y). \end{aligned}$$

Therefore  $(x, y)$  is a  $G_1 \times G_2$ -periodic point of  $f \times h$ . Hence the set of  $G_1 \times G_2$ -periodic points of  $f \times h$  is dense in  $X \times Y$ .  $\square$

**Definition 3.3.** Let  $(X, d)$  be a metric  $G$ -space and  $f : X \rightarrow X$  be continuous. If for every pair of non-empty open sets  $U, V$  of  $X$ , there exists an  $n_0 \in N$  and  $g \in G$  such that for all  $n \geq n_0$ ,  $(gf^n(U)) \cap V \neq \phi$  then  $f$  is called topologically  $G$ -mixing.

**Remark 3.4.** Clearly every topologically  $G$ -mixing map is topologically  $G$ -transitive.

**Theorem 3.5.** Let  $(X, d_1)$  be a metric  $G_1$ -space,  $(Y, d_2)$  be a metric  $G_2$ -space and  $f : X \rightarrow X$ ,  $h : Y \rightarrow Y$  be continuous maps. If  $f$  is topologically  $G_1$ -mixing and  $h$  is topologically  $G_2$ -mixing then  $f \times h$  is topologically  $G_1 \times G_2$ -mixing.

*Proof.* Let  $W_1$  and  $W_2$  be non-empty open sets in  $X \times Y$ . Then there exist open sets  $U_1, U_2$  in  $X$  and  $V_1, V_2$  open in  $Y$  such that  $U_1 \times V_1 \subset W_1$  and  $U_2 \times V_2 \subset W_2$ . By assumption there exists  $n_1, n_2 \in \mathbb{N}$  and  $g, k \in G$  such that

$$(gf^{n_1}(U_1)) \cap U_2 \neq \emptyset$$

and

$$(kh^{n_2}(V_1)) \cap V_2 \neq \emptyset$$

for all  $n \geq n_1$  and for all  $m \geq n_2$ . Let  $n_0 = \max\{n_1, n_2\}$ . Then for all  $n \geq n_0$ ,

$$\begin{aligned} & ((g, k)(f \times h)^n(U_1 \times V_1)) \cap (U_2 \times V_2) \\ &= ((g, k)(f^n(U_1) \times h^n(V_1))) \cap (U_2 \times V_2) \\ &= ((gf^n(U_1)) \cap U_2) \times ((kh^n(V_1)) \cap V_2) \\ &\neq \emptyset. \end{aligned}$$

Hence  $f \times h$  is topologically  $G_1 \times G_2$ - mixing.  $\square$

**Theorem 3.6.** *Let  $(X, d_1)$  be a metric  $G_1$ - space,  $(Y, d_2)$  be a metric  $G_2$ - space and  $f : X \rightarrow X, h : Y \rightarrow Y$  be equivariant maps. If  $f$  is Devaney's  $G_1$ - chaotic, topologically  $G_1$ - mixing and  $h$  is Devaney's  $G_2$ - chaotic, topologically  $G_2$ - mixing then  $f \times h$  is Devaney's  $G_1 \times G_2$ - chaotic.*

*Proof.* By Theorem 3.1,  $f \times h$  is  $G_1 \times G_2$ - sensitive dependent on initial conditions. By Theorem 3.2,  $f \times h$  has dense  $G_1 \times G_2$ - periodic points. By Theorem 3.5, the map  $f \times h$  is  $G_1 \times G_2$ - topologically mixing and hence  $G_1 \times G_2$ - is topologically transitive. Thus  $f \times h$  is Devaney's  $G_1 \times G_2$ - chaotic on  $X \times Y$ .  $\square$

**Remark 3.7.** Under trivial action of  $G$  on  $X$ , the above result coincides with Theorem 2.1 for chaotic maps .

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