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Some Applications of the Euler-Maclaurin Summation Formula

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Abstract

Let n be a fixed positive integer. In this article we obtain asymptotic formulae for $\sum_{j=1}^{N} j^n \log j$ and $\prod_{i=1}^{N} j^{(j^n)}$. For example if $n = 1$ we obtain the asymptotic formulae,

$$\sum_{i=1}^{N} j \log j = \frac{N^2}{2} \log N - \frac{N^2}{4} + \frac{1}{2} N \log N + \frac{1}{12} \log N + C_1 + o(1),$$

and

$$\prod_{i=1}^{N} j^j \sim e^{C_1} \ \frac{N^{\frac{N^2}{2}+\frac{N}{2}+\frac{1}{12}}}{e^{\frac{N^2}{4}}},$$

where C_1 is a constant.

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1 Introduction

Bernoulli numbers are among the most distinguished and important numbers in all of mathematics. Indeed, they play a vital role in number theory. The Bernoulli numbers can be defined in the following way (see [1], chapter V)

$$B_0 = 1$$

$$\sum_{i=0}^{n-1} \binom{n}{i} B_i = 0 \qquad (n \geq 2)$$

Using this relation we obtain the first Bernoulli numbers, namely $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, $B_5 = 0$.

It is well known (see [1], chapter V) that if $i \geq 3$ is odd then $B_i = 0$.

The n-th Bernoulli polynomial is defined in the following way

$$B_n(x) = \sum_{j=0}^{n} \binom{n}{j} B_j x^{n-j}.$$

The first few Bernoulli polynomials are

$$B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \quad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$$

In the following theorem we establish the Euler-Maclaurin summation formula.

Theorem 1.1 *Let $a < b$ be integers and let m a positive integer. If $f(x)$ has m continuous derivatives on the interval $[a, b]$, then*

$$\begin{aligned} \sum_{j=a+1}^{b} f(j) &= \int_a^b f(x)\, dx + \sum_{i=1}^{m} (-1)^i \frac{B_i}{i!} \left(f^{(i-1)}(b) - f^{(i-1)}(a) \right) \\ &+ \frac{(-1)^{m-1}}{m!} \int_a^b B_m\,(x - \lfloor x \rfloor)\, f^{(m)}(x)\, dx. \end{aligned} \tag{1}$$

Proof. See [1], chapter V.

Let n be a fixed positive integer. We shall need the following well-known formula

$$P_n(N) = \sum_{i=1}^{N} i^n = \frac{N^{n+1}}{n+1} + \frac{1}{2}N^n + \frac{1}{n+1} \sum_{i=2}^{n} \binom{n+1}{i} B_i N^{n+1-i} \qquad (N \geq 1).$$

For example, we have

$$P_1(N) = 1 + 2 + \cdots + N = \frac{N^2}{2} + \frac{N}{2},$$

$$P_2(N) = 1^2 + 2^2 + \cdots + N^2 = \frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}.$$

$$P_3(N) = 1^3 + 2^3 + \cdots + N^3 = \frac{N^4}{4} + \frac{N^3}{2} + \frac{N^2}{4}.$$

2 Preliminary Results

Let us consider the function

$$f(x) = f^{(0)}(x) = x^n \log x, \tag{2}$$

where n is a positive integer.

Let $f^{(k)}(x)$ be its k-th derivative. The following theorem holds.

Theorem 2.1 *We have*

$$f^{(k)}(x) = a_k x^{n-k} \log x + b_k x^{n-k} \qquad (k = 1, \ldots, n), \tag{3}$$

where

$$a_k = n(n-1)\cdots(n-(k-1)), \tag{4}$$

and

$$b_k = \sum_{i=0}^{k-1} \frac{n(n-1)\cdots(n-(k-1))}{n-i}. \tag{5}$$

Besides

$$f^{(n+1)}(x) = \frac{n!}{x}, \tag{6}$$

$$f^{(n+2)}(x) = -\frac{n!}{x^2}. \tag{7}$$

Proof. Clearly formulae (3), (4) and (5) are true if $k = 1$. This is the unique case if $n = 1$. If $n \geq 2$, suppose that formulae (3), (4) and (5) are true for k such that $1 \leq k \leq n-1$. That is

$$\begin{aligned} f^{(k)}(x) &= n(n-1)\cdots(n-(k-1))x^{n-k}\log x \\ &+ \left(\sum_{i=0}^{k-1} \frac{n(n-1)\cdots(n-(k-1))}{n-i}\right) x^{n-k}. \end{aligned}$$

If we derive then we find that

$$\begin{aligned} f^{(k+1)}(x) &= n(n-1)\cdots(n-(k-1))(n-k)x^{n-(k+1)}\log x \\ &+ n(n-1)\cdots(n-(k-1))x^{n-(k+1)} \\ &+ \left(\sum_{i=0}^{k-1} \frac{n(n-1)\cdots(n-(k-1))(n-k)}{n-i}\right) x^{n-(k+1)} \\ &= n(n-1)\cdots(n-k)x^{n-(k+1)}\log x \\ &+ \left(\sum_{i=0}^{k} \frac{n(n-1)\cdots(n-k)}{n-i}\right) x^{n-(k+1)}. \end{aligned}$$

That is, formulae (3), (4) and (5) are true for $k+1$. This proves formulae (3), (4) and (5).

If $k = n$ then equation (3) becomes (see (4) and (5))

$$f^{(n)}(x) = n! \log x + \sum_{i=0}^{n-1} \frac{n!}{n-i}. \tag{8}$$

Equations (6) and (7) are an immediate consequence of equation (8). The theorem is proved.

We shall need the following integral

$$\int x^n \log x \; dx = \frac{x^{n+1}}{n+1}\log x - \frac{1}{(n+1)^2}x^{n+1} + C. \tag{9}$$

3 Main Results

Lemma 3.1 *The integral*

$$\int_1^\infty B_{n+2}\left(x - \lfloor x \rfloor\right) \frac{1}{x^2}\, dx \qquad (n \geq 1) \tag{10}$$

is convergent.

Proof. Note that $0 \leq x - \lfloor x \rfloor < 1$. On the other hand (see (1), chapter V) $B_{n+2}(0) = B_{n+2}(1)$. Consequently $B_{n+2}(x - \lfloor x \rfloor)$ is continuous and with period 1 on the interval $(-\infty, \infty)$.

Therefore there exist $A > 0$ such that $\left|B_{n+2}\left(x - \lfloor x \rfloor\right)\right| \leq A$. Consequently we have

$$\left|B_{n+2}\left(x - \lfloor x \rfloor\right) \frac{1}{x^2}\right| \leq \frac{A}{x^2} \qquad (x \geq 1).$$

Now, the integral

$$\int_1^\infty \frac{A}{x^2}\, dx$$

is convergent. Therefore (comparison criterion) the integral

$$\int_1^\infty \left|B_{n+2}\left(x - \lfloor x \rfloor\right) \frac{1}{x^2}\right|\, dx \qquad (n \geq 1)$$

is also convergent. Thus, the integral (10) converges absolutely and hence converges. The lemma is proved.

The following theorem is our main theorem.

Theorem 3.2 *Let n be an arbitrary but fixed positive integer. The following asymptotic formula holds*

$$\sum_{j=1}^{N} j^n \log j = D_n(N) \log N - H_n(N) + C_n + o(1) \tag{11}$$

where

$$D_n(N) = P_n(N) + (-1)^{n+1} \frac{B_{n+1}}{(n+1)}, \quad H_n(N) = \frac{N^{n+1}}{(n+1)^2} - \sum_{i=2}^{n} \frac{B_i}{i!} b_{i-1} N^{n-(i-1)}.$$

and C_n is a constant depending of n. Note that $D_n(N)$ and $H_n(N)$ are polynomials in N of degree $n+1$.

Proof. We have (see (1) with $m = n+2$, (8), (6) , (7), (9) and lemma 3.1)

$$\begin{aligned}
\sum_{j=1}^{N} j^n \log j &= \sum_{j=2}^{N} j^n \log j = \int_1^N x^n \log x \, dx - B_1 \left(f^{(0)}(N) - f^{(0)}(1) \right) \\
&+ \sum_{i=2}^{n} (-1)^i \frac{B_i}{i!} \left(f^{(i-1)}(N) - f^{(i-1)}(1) \right) + (-1)^{n+1} \frac{B_{n+1}}{(n+1)!} \\
& (n! \log N) + (-1)^{n+2} \frac{B_{n+2}}{(n+2)!} \\
& \left(\frac{n!}{N} - n! \right) + \frac{(-1)^n}{(n+1)(n+2)} \\
& \int_1^N B_{n+2} \left(x - \lfloor x \rfloor \right) \frac{1}{x^2} \, dx = \frac{N^{n+1}}{n+1} \log N - \frac{N^{n+1}}{(n+1)^2} \\
&- B_1 f^{(0)}(N) + \sum_{i=2}^{n} (-1)^i \frac{B_i}{i!} f^{(i-1)}(N) + (-1)^{n+1} \frac{B_{n+1}}{(n+1)} \\
& \log N + C_n + o(1),
\end{aligned}$$

where C_n is a constant. That is

$$\begin{aligned}
\sum_{j=1}^{N} j^n \log j &= \frac{N^{n+1}}{n+1} \log N - \frac{N^{n+1}}{(n+1)^2} - B_1 f^{(0)}(N) + \sum_{i=2}^{n} (-1)^i \frac{B_i}{i!} f^{(i-1)}(N) \\
&+ (-1)^{n+1} \frac{B_{n+1}}{(n+1)} \log N + C_n + o(1). \qquad (12)
\end{aligned}$$

Substituting (2) and (3) into (12) we obtain

$$\begin{aligned}
\sum_{j=1}^{N} j^n \log j &= \frac{N^{n+1}}{n+1} \log N - \frac{N^{n+1}}{(n+1)^2} + \frac{1}{2} N^n \log N \\
&+ \sum_{i=2}^{n} (-1)^i \frac{B_i}{i!} \left(a_{i-1} N^{n-(i-1)} \log N + b_{i-1} N^{n-(i-1)} \right) \\
&+ (-1)^{n+1} \frac{B_{n+1}}{(n+1)} \log N + C_n + o(1) \qquad (13)
\end{aligned}$$

Since $B_1 = -1/2$ (see the introduction). Equation (13) can be written in the form

$$\sum_{j=1}^{N} j^n \log j = \left(A_n(N) + (-1)^{n+1} \frac{B_{n+1}}{(n+1)} \right) \log N - H_n(N) + C_n + o(1) \quad (14)$$

where

$$A_n(N) = \frac{N^{n+1}}{n+1} + \frac{1}{2} N^n + \sum_{i=2}^{n} \frac{B_i}{i!} a_{i-1} N^{n-(i-1)} \qquad (15)$$

and
$$H_n(N) = \frac{N^{n+1}}{(n+1)^2} - \sum_{i=2}^{n} \frac{B_i}{i!} b_{i-1} N^{n-(i-1)}. \tag{16}$$
Since if $i \geq 2$ then $(-1)^i B_i = B_i$ (see the introduction).

Substituting (4) into (15) we find that (see the polynomial $P_n(N)$ in the introduction)

$$\begin{aligned} A_n(N) &= \frac{N^{n+1}}{n+1} + \frac{1}{2}N^n + \sum_{i=2}^{n} \frac{B_i}{i!}(n(n-1)\cdots(n-(i-2)))N^{n-(i-1)} \\ &= \frac{N^{n+1}}{n+1} + \frac{1}{2}N^n + \frac{1}{n+1}\sum_{i=2}^{n} B_i \frac{(n+1)n\cdots(n-(i-2))}{i!} N^{n+1-i} \\ &= \frac{N^{n+1}}{n+1} + \frac{1}{2}N^n + \frac{1}{n+1}\sum_{i=2}^{n} \binom{n+1}{i} B_i N^{n+1-i} = P_n(N) \end{aligned} \tag{17}$$

Equations (14), (15), (16) and (17) give equation (11). The theorem is proved.

Theorem 3.3 *Let n be an arbitrary but fixed positive integer. The following asymptotic formula holds*

$$\prod_{j=1}^{N} j^{(j^n)} \sim e^{C_n} \frac{N^{D_n(N)}}{e^{H_n(N)}}, \tag{18}$$

Proof. It is an immediate consequence of equation (11). The theorem is proved.

Remark 3.4 *Note that equation (18) is a generalization of the Stirling's formula. Namely*
$$\prod_{j=1}^{N} j = N! \sim e^{C_0} \frac{N^{N+1/2}}{e^N},$$
where in this case $C_0 = \log\left(\sqrt{2\pi}\right)$.

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References

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