International Mathematical Forum, Vol. 7, 2012, no. 55, 2705 - 2714

Further Generalizations of Boolean Rings

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Abstract

McCoy and Montgomery [3] introduced the concept of a p-ring (p prime) as a ring R in which $x^p = x$ and $px = 0$ for all x in R. Thus, Boolean rings are simply 2-rings ($p = 2$). With this as motivation, we define a generalized p-ring as a ring R such that $x^p y - xy^p \in N$ for all x, y in $R \setminus (N \cup C)$, and $px = 0$ for all x in R, where N and C denote the set of nilpotents and center of R, respectively. We consider the commutativity behavior of generalized p-rings (p prime). In particular, we prove that a generalized p-ring (p prime) is commutative if and only if its idempotents are central and its nilpotents commute.

Mathematics Subject Classification: 16U80, 16D70

Keywords: generalized p-rings, Jacobson radical, subdirect sum, subdirectly irreducible ring, commutator ideal, p-like rings

Throughout, R is a ring, not necessarily with identity, N is the set of nilpotents, C is the center, and J is the Jacobson radical of R. As usual, $[x, y]$ will denote the commutator $xy - yx$.

Definition 1. *A ring R is called a generalized p-ring (p prime) if*

$$x^p y - xy^p \in N \text{ for all } x, y \in R \setminus (N \cup C), \text{ and } px = 0 \text{ for all } x \in R. \quad (1)$$

In preparation for the proofs of the main theorems, we need the following three lemmas.

Lemma 1 ([1]). *Suppose R is a ring in which each element x is central or potent in the sense that $x^k = x$ for some $k > 1$. Then R is commutative.*

Lemma 2. *Suppose R is a ring of prime characteristic p and with identity, and suppose that all idempotents of R are central. Then, for all b in R,*

$$b^p = b \text{ implies } b \text{ is central.} \quad (2)$$

Proof. Suppose $b^p = b$. Then b^{p-1} is idempotent, and hence by hypothesis b^{p-1} is central. Therefore, for all r in R,

$$b^{p-1}(rb - br) = b^{p-1}(rb) - b^p r = (rb)b^{p-1} - b^p r = rb^p - b^p r = rb - br,$$

and hence

$$(b^{p-1} - 1)(rb - br) = 0 \text{ for all } r \text{ in } R. \tag{3}$$

Since R is of *prime* characteristic p, an elementary number-theoretic result shows that (3) is equivalent to

$$(b+1)(b+2)\ldots(b+(p-1))(rb - br) = 0, \ (r \in R). \tag{4}$$

Moreover, since R is of *prime* characteristic p,

$$b^p = b \text{ implies } (b+1)^p = b+1,$$

and hence the above argument may be repeated with b replaced by $b+1$ throughout to obtain (see (4))

$$(b+2)(b+3)\ldots(b+(p-1))(b+p)(r(b+1) - (b+1)r) = 0,$$

and hence (since $b + p = b$)

$$b(b+2)(b+3)\ldots(b+(p-1))(rb - br) = 0. \tag{5}$$

Subtracting (5) from (4), we get

$$1 \cdot (b+2)(b+3)\ldots(b+(p-1))(rb - br) = 0. \tag{6}$$

Repeating this argument, where b is replaced by $(b+1)$ again throughout, we obtain (see (6))

$$1 \cdot (b+3)(b+4)\ldots(b+(p-1))(b+p)(rb - br) = 0,$$

and hence (since $b + p = b$)

$$1 \cdot b \cdot (b+3)(b+4)\ldots(b+(p-1))(rb - br) = 0. \tag{7}$$

Subtracting (7) from (6), we obtain

$$1 \cdot 2 \cdot (b+3)(b+4)\ldots(b+(p-1))(rb - br) = 0.$$

Continuing this process, we eventually obtain

$$1 \cdot 2 \cdot 3 \ldots (p-1)(rb - br) = 0. \tag{8}$$

Since $(p-1)!$ is relatively prime to the prime characteristic p of R, we see that (8) implies $rb - br = 0$ for all r in R, which proves the lemma. □

Lemma 3. *Suppose R is a generalized p-ring (p prime), and suppose all idempotents are central. Then, $N \subseteq J$.*

Proof. Let $a \in N, x \in R$. If $ax \in N$, then ax is right quasi regular (r.q.r.). Also, if $ax \in C$, then $ax \in N$, and hence again ax is r.q.r. Moreover, if $(ax)^2 \in N$, then $ax \in N$, which implies that ax is r.q.r. Finally, suppose $(ax)^2 \in C$ and $a^k = 0$ (since $a \in N$). Then, $((ax)^2)^k = (ax)^2(ax)^2 \ldots (ax)^2 = a^k y = 0$ for some $y \in R$. Therefore, $ax \in N$, and hence again ax is r.q.r. The only case left to consider is $ax \notin (N \cup C)$ and $(ax)^2 \notin (N \cup C)$, which implies by (1) that $(ax)^p(ax)^2 - (ax)(ax)^{2p} \in N$. So $(ax)^{p+2} - (ax)^{2p+1} \in N$. Since $p+2 \neq 2p+1$, this implies that

$$(ax)^q = (ax)^{q+1} g(ax), \text{ for some } g(\lambda) \in \mathbb{Z}[\lambda].$$

Hence, $(ax)^q = (ax)^q(axg(ax)) = (ax)^q(axg(ax))^2 = \ldots = (ax)^q(axg(ax))^q$. It is readily verified that $(axg(ax))^q$ is idempotent, and hence

$$(ax)^q = (ax)^q e, \ e^2 = e \in aR, \ e = (axg(ax))^q. \tag{9}$$

Note that for some $r \in R$,

$$e = e^2 = ee = e(ar) = aer \text{ (since } e \text{ is central).}$$

By re-iterating, we see that

$$e = aer = a^2 er^2 = \ldots = a^k er^k \text{ for all integer } k.$$

Since $a \in N$, let $a^k = 0$. Then the above equalities imply that $e = 0$, and hence by (9), $(ax)^q = 0$, which implies that $ax \in N$, and hence ax is r.q.r. The net result is that ax is r.q.r. for all $a \in N, x \in R$, and hence $a \in J$. This proves the lemma. □

We are now in a position to prove our main theorems.

Theorem 1. *A generalized p-ring (p prime) R with central nilpotents is commutative.*

Proof. Since $N \subseteq C$, $N \cup C = C$, and hence (1) becomes

$$x^p y - xy^p \in N \text{ for all } x, y \in R \setminus C. \tag{10}$$

Suppose $x \notin C$. We distinguish two cases.
Case 1. $x^2 \notin C$. Then, $x \notin C$ and $x^2 \notin C$, and hence by (10), $x^p(x^2) - x(x^2)^p \in N$ which implies $x^{p+2} - x^{2p+1} \in N$. Thus,

$$(x - x^p)^{p+2} = (x - x^p) \cdot x^{p+1} g(x) = (x^{p+2} - x^{2p+1}) g(x) \in N.$$

So $x - x^p \in N \subseteq C$, and hence $x - x^p \in C$ (if $x^2 \notin C$).
Case 2. $x^2 \in C$. Then $x - x^2 \notin C$ (since $x \notin C$). Moreover, $x \notin C$, by hypothesis, which implies by (10),

$$x^p(x - x^2) - x(x - x^2)^p \in N.$$

Therefore, $x^{p+1} - x^{p+2} - x(x^p - x^{2p}) \in N$ (since R is of characteristic p, p prime), and thus $x^{p+2} - x^{2p+1} \in N$, which as we saw in Case 1, implies $x - x^p \in N \subseteq C$. So $x - x^p \in C$. The net result is: $x \notin C$ implies $x - x^p \in C$. Thus, $x - x^p \in C$ for all $x \in R$, which implies by a well known theorem of Herstein [2], that R is commutative. □

Corollary 1. *A reduced generalized p-ring is commutative.*

Corollary 2. *A p-ring is commutative.*

Theorem 2. *Suppose R is a generalized p-ring (p prime). Then,*

(i) $x \notin C$ *implies* $x - x^p \in N$.

(ii) $x \notin C$ *implies* $x^q = x^q e$, $q \geq 1$, $e^2 = e \in x\mathbb{Z}[x]$.

(iii) Every subring and every homomorphic image of a generalized p-ring is also a generalized p-ring.

Proof. (i) Suppose $x \notin C$ and $x - x^p \notin N$. Then $x \notin N$, $x \notin C$. We distinguish two cases.

Case 1. $x^2 \notin C$. Then $x^2 \notin N$ and $x^2 \notin C$. Also, $x \notin N$ and $x \notin C$, and hence by (1), $x^p(x^2) - x(x^2)^p \in N$, and hence as in the proof of Theorem 1, $x - x^p \in N$, contradiction.

Case 2. $x^2 \in C$. Then $x - x^2 \notin C$ (since $x \notin C$). If $x - x^2 \in N$, then $(x - x^2) + x(x - x^2) + \ldots + x^{p-2}(x - x^2) \in N$, and thus $x - x^p \in N$, contradiction. This contradiction proves that $x - x^2 \notin N$. The net result is: $x - x^2 \notin (N \cup C)$ and $x \notin (N \cup C)$, and hence by (1),

$$x^p(x - x^2) - x(x - x^2)^p \in N.$$

Thus, as we saw in the proof of Theorem 1, Case 2, $x - x^p \in N$, contradiction. These two contradictions in these two cases prove part (i).

(ii) By part (i), $x - x^p \in N$, and hence $(x - x^p)^q = 0$ for some $q \geq 1$, which implies that $x^q = x^{q+1}g(x)$, $g(x) \in \mathbb{Z}[x]$. Thus, $x^q = x^q(xg(x)) = x^q(xg(x))^2 = \ldots = x^q(xg(x))^q = x^q e$, where $e = (xg(x))^q$ is (as is readily verified) idempotent, proving (ii).

(iii) Follows at once from Definition 1. □

Theorem 3. *A generalized p-ring with identity is commutative if and only if* $E \subseteq C$ *and* $N \cap J$ *is commutative, where E denotes the set of idempotents.*

Proof. Clearly a commutative ring satisfies the two given conditions on E and $N \cap J$. To prove the converse, suppose that $E \subseteq C$ and $N \cap J$ is commutative. By Lemma 3, $N \subseteq J$, and hence $N = N \cap J$ is commutative. Thus,

$$N \text{ is commutative.} \tag{11}$$

We claim that

$$N \subseteq C. \tag{12}$$

The proof is by contradiction. Suppose not. Then,

$$\text{for some } a \in N,\ x \in R,\ [a, x] \neq 0. \tag{13}$$

In view of (11) and (13), $x \notin N$ and $x \notin C$. Also, $[a, x + 1] = [a, x] \neq 0$, and hence, since $a \in N$, $x+1 \notin (N \cup C)$. Therefore, by (1), $x^p(x+1) - x(x+1)^p \in N$, which implies that $x - x^p \in N$ (since R is of prime characteristic p). Thus,

$$x - x^p \in N. \tag{14}$$

Since $x - x^p \in N$, $(x - x^p)^{p^k} = 0$ for some positive integer k, which implies (since R is of prime characteristic p)

$$x^{p^k} = x^{p^{k+1}}. \tag{15}$$

Moreover, by (14),

$$(x - x^p) + (x - x^p)^p + (x - x^p)^{p^2} + \ldots + (x - x^p)^{p^{k-1}} \in N,$$

and hence $x - x^{p^k} \in N$. Also, by (15), $(x^{p^k})^p = (x^{p^k})$. So

$$x - x^{p^k} \in N \text{ and } (x^{p^k})^p = (x^{p^k}). \tag{16}$$

Moreover, $x^{p^k} \in C$, by Lemma 2 (see (16)), and hence

$$[a, x] = [a, (x - x^{p^k}) + x^{p^k}] = [a, x - x^{p^k}] = 0,$$

by (16) and (11). Therefore, $[a, x] = 0$, which contradicts (13). This contradiction proves (12). The theorem now follows from (12) and Theorem 1. □

Corollary 3. *A generalized p-ring with identity and with central idempotents and commuting nilpotents is commutative.*

The next two lemmas will be needed in order to drop the hypothesis that $1 \in R$ in Theorem 3.

Lemma 4. *Let R be a generalized p-ring, and suppose $\sigma : R \to S$ is a homomorphism of R onto S. Then the set N^* of nilpotent elements of S is contained in $\sigma(N) \cup C^*$, where C^* is the center of S and N is the set of nilpotents of R.*

Proof. Suppose $s \in N^*$, and $s \notin \sigma(N) \cup C^*$, and suppose d is a preimage of s; that is, $\sigma(d) = s$. Then $d \notin C$ and $d \notin N$. By Theorem 2(i), $d - d^p \in N$. Since $s \in N^*$, $s^q = 0$ for some $q \geq 1$. Note that $(d - d^p) + d^{p-1}(d - d^p) + \ldots + (d^{p-1})^q(d - d^p) = d - d^{(p-1)q+p} \in N$, since this is a sum of pairwise commuting nilpotents. This implies that

$$\sigma(d - d^{(p-1)q+p}) \in \sigma(N),$$

and hence $s - s^{(p-1)q+p} \in \sigma(N)$, which shows that $s \in \sigma(N)$, (recall that $s^q = 0$), which contradicts the hypothesis that $s \notin \sigma(N)$. This contradiction proves the lemma. □

Lemma 5. *Suppose R is a generalized p-ring with central idempotents, and suppose $\sigma : R \to R_i$ is a homomorphism of R onto a subdirectly irreducible ring R_i. Then either (a) Each element of R_i is central or nilpotent or (b) Each element of R_i is central or nilpotent or a unit in R_i.*

Proof. By Theorem 2(ii), if $x \in R$, then $x \in C$ or $x^q = x^q e$, $q \geq 1$, $e^2 = e \in x\mathbb{Z}[x]$. By hypothesis, $e \in C$, and hence we conclude that

$$\begin{aligned} x_i \in R_i \text{ implies } x_i \text{ is central or } x_i^q = x_i^q e_i,\ q \geq 1, \\ e_i^2 = e_i,\ e_i \in x_i\mathbb{Z}[x_i],\ (e_i = \sigma(e)),\ e_i \text{ central.} \end{aligned}$$

If R_i does not have an identity then $e_i = 0$, and hence each element of R_i is central or nilpotent. If R_i has an identity, then $e_i = 0$ or $e_i = 1$, and hence each element of R_i is central or nilpotent or a unit in R_i, since $e_i \in x_i\mathbb{Z}[x_i]$. This proves the lemma. □

We are now in a position to drop the hypothesis that R has an identity in Theorem 3.

Theorem 4. *Any generalized p-ring R is commutative if and only if $E \subseteq C$ and $N \cap J$ is commutative.*

Proof. Suppose $E \subseteq C$ and $N \cap J$ is commutative. As is well known, $R \cong$ a subdirect sum of subdirectly irreducible rings R_i $(i \in \Gamma)$. Let $\sigma_i : R \to R_i$ be the natural homomorphism. By Lemma 5, either $R_i = N_i \cup C_i$ or $N_i \cup C_i \cup U_i$, where N_i, C_i, U_i denote the set of nilpotents, the center, and the set of units in R_i. Moreover, by Lemma 4, $N_i \subseteq \sigma_i(N) \cup C_i$. Furthermore, by Lemma 3, $N \subseteq J$, and hence $N = N \cap J$, and thus N is commutative (since $N \cap J$ is commutative). Since $N_i \subseteq \sigma_i(N) \cup C_i$, it follows that N_i is commutative also. The result is:

(a) $R_i = N_i \cup C_i$, N_i commutative, or

(b) $R_i = N_i \cup C_i \cup U_i$, $U_i =$ set of units in R_i.

We claim that in case (b), $U_i \subseteq C_i$. The proof is by contradiction. Suppose not, and let $u_i \in U_i$ be such that $[u_i, x_i] \neq 0$. Let d be a preimage of u_i; that is, $\sigma_i(d) = u_i$. Then $d \notin C$, and hence by Theorem 2(ii), $d^q = d^q e$, $e^2 = e$, $e \in R$. Therefore, $u_i^q = u_i^q \sigma_i(e)$, which implies that $\sigma_i(e) = 1$, $e \in C$ (since u_i is a unit). Hence, eR is a ring *with identity* e. Moreover, eR satisfies all the hypotheses of Theorem 3. (In verifying this, keep in mind that $J(eR) = eJ(R) \subseteq J(R)$ and $N(eR) \subseteq N(R)$, and hence $N(eR) \cap J(eR) \subseteq N(R) \cap J(R)$, which implies that $N(eR) \cap J(eR)$ is commutative, since, by hypothesis, $N(R) \cap J(R) = N \cap J$ is commutative.) Therefore, by Theorem 3, eR is commutative. Let $x_i, y_i \in R_i$, and let $\sigma_i(x) = x_i$, $\sigma_i(y) = y_i$, $x, y \in R$. Since eR is commutative, $[ex, ey] = 0$, which implies $[\sigma_i(ex), \sigma_i(ey)] = 0$, and hence $[\sigma_i(e)\sigma_i(x), \sigma_i(e)\sigma_i(y)] = 0$. So $[\sigma_i(x), \sigma_i(y)] = 0$, (since $\sigma_i(e) = 1$), and hence $[x_i, y_i] = 0$ for all $x_i, y_i \in R_i$; that is, R_i is commutative, a contradiction (since $[u_i, x_i] \neq 0$). This contradiction proves that $U_i \subseteq C_i$, and hence $N_i \subseteq C_i$ (since $a_i \in N_i$ implies $1 + a_i \in U_i \subseteq C_i$), which implies that R_i is commutative, by Theorem 1 and Theorem 2(iii) (in case (b)).

Returning to case (a), we have $R_i = N_i \cup C_i$, and N_i is commutative, which readily implies that R_i is commutative, and hence R itself is commutative, which proves the theorem. □

Closely related to commutativity is the notion of a ring having a nil commutator ideal. In this connection, we have the following theorem.

Theorem 5. *Suppose R is a generalized p-ring with central idempotents. Then the commutator ideal of R is nil.*

Proof. By Lemma 3, $N \subseteq J$. We now prove that

$$J \subseteq N \cup C \tag{17}$$

Suppose $j \in J$, $j \notin C$. Then, by Theorem 2(ii), for some $q \geq 1$,

$$j^q = j^q e,\ e^2 = e \in j\mathbb{Z}[j].$$

Since e is an idempotent in J, $e = 0$, and hence $j^q = j^q \cdot 0 = 0$, which implies that $j \in N$, proving (17).

Next we prove that

$$N \text{ is an ideal.} \tag{18}$$

By (17), and the fact that $N \subseteq J$, we get $N \subseteq J \subseteq N \cup C$. Suppose $a \in N$, $x \in R$. Then $a \in J$, $x \in R$, and hence $ax \in J$, which implies that $ax \in N \cup C$

(by (17)). Hence, $ax \in N$ or $ax \in C$ (which implies that $ax \in N$). So $ax \in N$. Similarly, $xa \in N$. Next, suppose $a \in N$, $b \in N$. Then $a \in J$, $b \in J$ (since $N \subseteq J$), and hence $a - b \in J \subseteq N \cup C$ (by (17)). So $a - b \in N$ or $a - b \in C$. If $a - b \in C$ then a commutes with b, and hence $a - b \in N$ again, which proves (18).

Returning to the ring R/N, note that, by Theorem 2(iii), R/N is a generalized p-ring, and hence by Theorem 2(i) (applied now to R/N), we see that every element of R/N is central or potent. Therefore, by a theorem of Bell (Lemma 1), R/N is commutative, which implies that the commutator ideal of R is nil, and the theorem is proved. □

We now consider a certain subclass of generalized p-rings (p prime).

Definition 2. *A* p-like *ring (p prime) is a ring R such that $x^p y = xy^p$ for all $x, y \in R \setminus (N \cup C)$, and $px = 0$ for all $x \in R$.*

Theorem 6. *Suppose R is a p-like ring (p prime) with identity. Then R is commutative.*

Proof. We first prove that

$$N \text{ is commutative.} \tag{19}$$

Suppose that $a \in N$, u is a unit in R. Our goal is to prove that

$$[a, u] = 0, \ (a \in N, \ u \text{ a unit}). \tag{20}$$

Since $a \in N$, there exists a *minimal* positive integer σ_0 such that

$$[a^\sigma, u] = 0 \text{ for all } \sigma \geq \sigma_0, \ \sigma_0 \textit{ minimal}. \tag{21}$$

We claim that $\sigma_0 = 1$. Suppose not. The *minimality* of σ_0 in (21), and the fact that $u \notin C$ (see (20)), show that $1 + a^{\sigma_0 - 1} \notin (N \cup C)$ and $u \notin (N \cup C)$, and hence by Definition 2,

$$(1 + a^{\sigma_0 - 1})^p u = (1 + a^{\sigma_0 - 1}) u^p,$$

which implies that (since both $1 + a^{\sigma_0 - 1}$ and u are units)

$$(1 + a^{\sigma_0 - 1})^{p-1} = u^{p-1}.$$

Thus,

$$[(1 + a^{\sigma_0 - 1})^{p-1}, u] = 0,$$

which implies (see (21))

$$[1 + (p-1)a^{\sigma_0 - 1}, u] = 0.$$

Therefore,

$$(p-1)[a^{\sigma_0 - 1}, u] = 0,$$

which implies that $[a^{\sigma_0-1}, u] = 0$ (since R is of characteristic p), contradicting the minimality of σ_0 (see (21)). This contradiction shows that $\sigma_0 = 1$, and hence by (21), $[a, u] = 0$, which proves (20). Now, let $b \in N$. Then, $1 + b$ is a unit, and hence by (20), $[a, 1 + b] = 0$ (since $a \in N$), which implies $[a, b] = 0$ for all $a, b \in N$, proving (19). Our next goal is to prove that

$$\text{All idempotents are central.} \tag{22}$$

Let $e^2 = e$, $x \in R$, $a = ex - exe$. Suppose $a \neq 0$. Then, $a \notin C$ (since $a \in C$ implies $ea = ae$, and hence $a = 0$, contradiction). Since $a \notin C$ and $a \in N$, $1 + a \notin (N \cup C)$. Moreover, since $a \neq 0$, $e \notin (N \cup C)$ either, and hence by Definition 2,

$$e^p(1 + a) = e(1 + a)^p = e(1 + a^p) = e + ea^p = e,$$

which implies that $e(1 + a) = e$, and thus $a = 0$, contradiction. Hence, $a = 0$; that is, $ex = exe$. Similarly, $xe = exe$, proving (22). The theorem now follows from (19), (22), and Theorem 4. □

Our final result is the following:

Theorem 7. *Suppose R is a p-like ring with central idempotents. Then, R is isomorphic to a subdirect sum of subdirectly irreducible rings R_i, where R_i is either nil or commutative.*

Proof. As is well known, $R \cong$ a subdirect sum of subdirectly irreducible rings R_i. A careful examination of the proof of Lemma 5 shows that if R_i does not have an identity, then $R_i = N_i \cup C_i$, where N_i and C_i are the set of nilpotents and center of R_i, respectively, which readily implies $R_i = N_i$ or $R_i = C_i$. On the other hand, if $1 \in R_i$, then by Theorem 6 and Theorem 2(iii), R_i is commutative, which proves the theorem. □

We conclude with the following remarks.

Remark 1. Theorems 3 and 4 are not true if in (1) of Definition 1 the two primes used there are different, as can be seen by taking

$$R = \left\{ \begin{bmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{bmatrix} : \ a, b, c \in GF(4) \right\}.$$

Note that $x^7y - xy^7 = 0$ for all $x, y \in R \setminus (N \cup C)$, and $2x = 0$ for all x in R. Moreover, all idempotents of R are central and N is commutative. But R is not commutative.

Remark 2. Theorem 6 is not true if we delete the hypothesis that $1 \in R$, as can be seen by taking

$$R = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} :\ 0, 1 \in GF(2) \right\}.$$

Note that $x^2y - xy^2 = 0$ for all $x, y \in R \setminus (N \cup C)$, and $2x = 0$ for all $x \in R$. But R is not commutative.

Related work appears in [4].

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Received: June, 2012