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Monads on a Multiprojective Space, $\mathbf{P}^a \times \mathbf{P}^b$

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Abstract. For all integers $a, b > 0$ we establish explicitly the existence of monads on a multiprojective Space $\mathbf{P}^a \times \mathbf{P}^b$ following the conditions established by Floystad. That is for all positive integers α, β, γ there exists a monad on the multiprojective space $X = \mathbf{P}^a \times \mathbf{P}^b$ whose maps A and B have entries being linear in two sets of homogeneous coordinates $x_0 : ... : x_a$ and $y_0 : ... : y_b$ and it takes the form:

$$0 \longrightarrow \mathcal{O}_X^{\alpha}(-1,-1) \xrightarrow{A} \mathcal{O}_X^{\beta} \xrightarrow{B} \mathcal{O}_X^{\gamma}(1,1) \longrightarrow 0$$

where the maps A and B are matrices with $B \cdot A = 0$ and they are of maximal rank.

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1. introduction

Monads appear in many contexts within algebraic geometry and they are very useful in construction of vector bundles with prescribed invariants like rank, determinants etc. Hopefully we shall give more results in this regard in the near future but for now, here we use a similar construction as Floystad in [2] we construct a monad on a multiprojective space via the segre embedding on $\mathcal{O}_X(1,1)$ which is very ample. Floystad gave an if and only theorem on the existence of the Monad and the conditions for the existence.

Definition 1.1. Let X be a smooth projective variety. A monad on X is a complex of vector bundles:

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

with α injective and β surjective.

Definition 1.2. [1].
Let X be a nonsingular projective variety over an algebraically closed field κ with characteristic 0. Let L be a very ample invertible sheaf and V, W, U be finite dimensional κ-vector spaces. A linear monad on X is a complex of sheaves,

$$0 \longrightarrow V \otimes L^{-1} \xrightarrow{\alpha} W \otimes \mathcal{O}_X \xrightarrow{\beta} U \otimes L \longrightarrow 0$$

where $\alpha \in Hom(V,W) \otimes H^0L$ and is injective and $\beta \in Hom(W,U) \otimes H^0L$ and is surjective.

The existence and classification of Linear Monads was given by Floystad in [2].

Lemma 1.3. *Let $k \geq 1$. There exists monads on $\mathbf{P}^k$ whose maps are matrices of linear forms,*

$$0 \longrightarrow \mathcal{O}^a_{\mathbf{P}^k}(-1) \xrightarrow{\alpha} \mathcal{O}^b_{\mathbf{P}^k} \xrightarrow{\beta} \mathcal{O}^c_{\mathbf{P}^k}(1) \longrightarrow 0$$

if and only if atleast one of the following is fulfilled;
(1) $b \geq 2c + k - 1$, $b \geq a + c$ and
(2) $b \geq a + c + k$

Proof. [2] □

Remark 1. This remark is a quick sketch which helps us demonstrate our proof below.
Define two matrices Z, T matrices of orders r by $r + n$ and r by $r + m$ respectively and denote them by;

$$Z = \begin{pmatrix} z_0 & z_1 \cdots & z_n & & & \\ & z_0 & z_1 \cdots & z_n & & \\ & & \ddots & & \ddots & \\ & & & z_0 & z_1 \cdots & z_n \end{pmatrix}$$

and

$$T = \begin{pmatrix} t_0 & t_1 \cdots & t_m & & & \\ & t_0 & t_1 \cdots & t_m & & \\ & & \ddots & & \ddots & \\ & & & t_0 & t_1 \cdots & t_m \end{pmatrix}$$

Floystad used the above matrices to construct the maps α β, matrices of linear forms on $\mathbf{P}^N$, where $N = m + n + 1$, the homogeneous coordinate being

$\kappa[z_0 : ... : z_n : t_0 : ... : t_m]$ and the maps are represented by R and S below

$$R = \left(\begin{array}{cccccc|cccccc} z_0 \cdots & z_n & & & & & t_0 \cdots & t_m & & & & \\ & \ddots & \ddots & & & & & \ddots & \ddots & & & \\ & & & z_0 \cdots & z_n & & & & & t_0 \cdots & t_m & \end{array} \right)$$

an r by $2r + m + n$ matrix and

$$S = \left(\begin{array}{cccc} t_0 \cdots & t_m & & \\ & \ddots & \ddots & \\ & & t_0 \cdots & t_m \\ \hline -z_0 \cdots & -z_n & & \\ & \ddots & \ddots & \\ & & -z_0 \cdots & -z_n \end{array} \right)$$

a $2r + m + n$ by $r + n + m$ matrix.

Setting $a = r + n + m$, $c = r$ and $b = 2r + n + m$ then $R \cdot S = 0$ and rank of $R = r$ from the Lemma above.

Lemma 1.4. $[Z_{r,r+n}] \cdot [T_{r+n,r+n+m}] = [T_{r,r+m}] \cdot [Z_{r+m,r+m+n}]$

Proof. [3] □

Moreover, the construction of R and S from Z and T is as follows:

$$R = \left(\; Z_{r,r+n} | T_{r,r+m} \; \right)$$

and

$$S = \left(\begin{array}{c} T_{r+n,r+n+m} \\ -Z_{r+m,r+n+m} \end{array} \right)$$

2. The Theorem

Theorem 2.1. *Let $\alpha, \beta, \gamma, a, b$ be positive integers such that at least one of the following conditions is true*
(1)$\beta \geq 2\gamma + N - 1$, and $\beta \geq \alpha + \gamma$,
(2)$\beta \geq \alpha + \gamma + N$, where $N = (a + 1)(b + 1) - 1$
then there exists a linear monad on $X = \mathbf{P}^a \times \mathbf{P}^b$ of the form;

$$0 \longrightarrow \mathcal{O}_X^{\alpha}(-1,-1) \xrightarrow{A} \mathcal{O}_X^{\beta} \xrightarrow{B} \mathcal{O}_X^{\gamma}(1,1) \longrightarrow 0$$

Proof. Define a segre embedding;
$\varphi : \mathbf{P}^a \times \mathbf{P}^b \hookrightarrow \mathbf{P}^N$ by

$$(x_0 : ... : x_a)(y_0 : ... : y_b) \longmapsto (x_0 y_0 : ... : x_a y_0 : ... : x_a y_b)$$

Now for all positive integers a, b there exists positive integers m, n such that $m + n + 1 = (a + 1)(b + 1) - 1$ and so there is a 1-1 and onto correspondence between $\{z_0 : ... : z_n : t_0 : ... : t_m\}$and $\{x_0y_0 : ... : x_ay_0 : ... : x_ay_b\}$ and so we are able to define the maps of the Monad we are constructing whose entries are polynomials of bidegree (1,1) linear on the x_is, $i = 1 \cdots, a$ and on y_js, $j = 1 \cdots, b$.

There are two cases to consider:
$(i)\beta = 2\gamma + N - 1$, and $\beta = \alpha + \gamma$, and
$(ii)\beta > 2\gamma + N - 1$, and $\beta > \alpha + \gamma$
Suppose case (i) is true then taking $\gamma = r$ then $\beta = 2r + m + n$, since $N - 1 = m + n$ and $\alpha = r + m + n$ then we have constructed our Monad since the maps A and B inherit the Flostad conditions. Note that it suffices to have considered case (i).

More specifically,we will have the matrices A and B taking the following forms which are basically the form of the matrices of Floystad, R and S above.

$$B = \left(\begin{array}{cccc|cccc} x_0y_0\cdots & x_iy_j & & & x_{i'}y_{j'}\cdots & x_ay_b & & \\ & \ddots & \ddots & & & \ddots & \ddots & \\ & & x_0y_0\cdots & x_iy_j & & & x_{i'}y_{j'}\cdots & x_ay_b \end{array}\right)$$

an r by $2r + m + n$ matrix just like R above and

$$A = \left(\begin{array}{cccc} x_{i'}y_{j'}\cdots & x_ay_b & & \\ & \ddots & \ddots & \\ & & x_{i'}y_{j'}\cdots & x_ay_b \\ \hline -x_0y_0\cdots & -x_iy_j & & \\ & \ddots & \ddots & \\ & & -x_0y_0\cdots & -x_iy_j \end{array}\right)$$

a $2r + m + n$ by $r + n + m$ matrix just like S.

Which we identify exactly with the Floystad coordinates $\{z_0 : ... : z_n : t_0 : ... : t_m\}$ as follows:
$z_0 = x_0y_0$, $t_m = x_ay_b$ and
$z_n = x_iy_j$, with j and i are defined as follows
$j = n \mod b + 1$ and $i = \mu - 1$,
where μ is a positive integer such that $(\mu - 1)(b + 1) \leq n < \mu(b + 1)$
and $t_0 = x_{i'}y_{j'}$ with j' and i' are defined as follows
$j' = n + 1 \mod b + 1$ and $i' = \lambda - 1$,
where λ is a positive integer such that $(\lambda - 1)(b + 1) \leq n + 1 < \lambda(b + 1)$
from which we can say that $B \cdot A = 0$ and rank of $B = r$ hence the theorem . $\square$

References

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