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A Relation between e – Rim and First Mullineux Partition

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To Prof. Meinolf GECK for 50th anniversary

Abstract

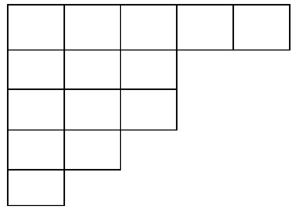
Mullineux in 1979 introduced the partition μ^{Δ} and Fayers in 2009 found the parallel partition μ^{∇} of the first partition μ^{Δ}, in general $\mu^{\Delta} \neq \mu^{\nabla}$. We considered in [8] and [9] when $\mu^{\Delta} = \mu^{\nabla}$. The main purpose of this work is to find an equal relation between the partition of (e – rim) and the partition of μ^{Δ} through mathematical and computer proof .

Mathematics subject classification: 05E10 and 20C30

Keywords: Partition theory, Mullineux partition, e – rim .

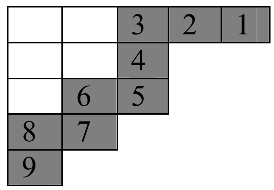
1 Introduction

Let r be a non – negative integer . A partition $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ of r is a sequence of non – negative integers such that $|\mu| = \sum_{j=1}^{n} \mu_j = r$ and $\mu_j \geq \mu_{j+1}$, $\forall\, j \geq 1$. The Young diagram of μ; denoted by $[\mu]$, is a left – justified array of squares with μ_j squares in the jth row. For example, if $\mu = (5,3,3,2,1)$. Then

$[\mu] =$ 

Ford and Kleshchev in [5] introduced the following definition :
The rimof a Young diagram of $[\mu]$ satisfies the following condition :
"A squares in row x and column y of $[\mu]$ belongs to its rim if and only if the square in row x+1 and column y+1 does not belong to $[\mu]$ " .

From the above example , the rim of (5,3,3,2,1) is



2 Mullineux's partitions

Fayers in [3] introduced the conditions for two partitions of mullineux, and he called for μ^{Δ} is the first Mullineux's partition and for μ^{∇} is the second Mullineux's partition. In general $\mu^{\Delta} \neq \mu^{\nabla}$.

Mahmood and Ghanem in [8] and [9], Found the condition under which $\mu^{\Delta} = \mu^{\nabla}$.

(2.1) <u>Diagram (A)</u> : [4]

Choose an integer b greater than or equal the number of parts of a partition , and define $\beta_i = \mu_i + \mathrm{b} - i$ for $1 \leq i \leq \mathrm{b}$. The set $\{\beta_1, \beta_2, \ldots, \beta_b\}$ is said to be a set of $\beta -$ number for . For example , if $\mu = (5,3,3,2,1)$ then $\beta -$ number are $\{9,6,5,3,1\}$ if we choose b = 5 and {10,7,6,4,2,1} if b = 6 .

We can represent $\beta -$ number by many runners depending on $e \geq 2$; we denote this diagram by (A), as follows :

$$\left.\begin{array}{cccc} \underline{\text{run. 1}} & \underline{\text{run. 2}} & \underline{\cdots} & \underline{\text{run. e}} \\ 0 & 1 & \cdots & \text{e - 1} \\ \text{e} & \text{e} + 1 & \cdots & \text{2e - 1} \\ \text{2e} & \text{2e} + 1 & \cdots & \text{3e - 1} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{array}\right\} \ldots (\text{A}) .$$

Where every $\beta -$ number will be represented by a bead which takes its location in diagram (A). From the above example,

e = 2 , b = 5	e = 2 , b = 6
– • – • – • • – – • – –	• – • – • – • • – – • –

e = 3 , b = 5	e = 3 , b = 6	e = 3 , b = 7
– • – • – • • – – • – –	• – • – • – • • – – • –	• • – • – • – • • – – •

Since the value of b $\geq$ n, then we deal with an infinite numbers of values of b. Here we want to mention that these values have a special diagram (A) for it, but there is a repeated part of this diagram with other values where a "Down – shifted" or "Up – shifted", occurs when we take the following :

b_1 if $b = n$, b_2 if $b = n + 1, \ldots, b_e$ if $b = n + (e - 1)$

<u>Definition (2.2)</u> : [7]

For any $\beta -$ numbers in diagram (A) , the values of $b_1, b_2, \ldots, b_e$ are called the guides .

We will define any diagram (A) that corresponds any b guides as a "main diagram" or "guide diagram" .

<u>Theorem (2.3)</u> : [7 Theorem 2.5]

There is e of main diagrams for any partition μ of r .

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The main diagrams from the above example where e = 3 are

$\mu = (5,3,3,2,1)$		
$b_1 = 5$	$b_1 + e$	$b_1 + 2e$
– • – • – • • – – • – –	• • • – • – • – • • – – • – –	• • • • • • – • – • – • • – – • – –
$b_2 = 6$	$b_2 + e$	$b_2 + 2e$
• – • – • – • • – – • –	• • • • – • – • – • • – – • –	• • • • • • • – • – • – • • – – • –
$b_3 = 7$	$b_3 + e$	$b_3 + 2e$
• • – • – • – • • – – •	• • • • • – • – • – • • – – •	• • • • • • • • – • – • – • • – – •

Fayers in [2] defined the following definitions :
"Given any partition μ, the conjugate partition μ is defined by $\mu'_t = \{j \geq 1 | \mu_j \geq t\}$, the partition μ is e – regular if there no exist $j \geq 1$ such that $\mu_j = \mu_{j+e-1} > 0$, also μ is e – restricted if $\mu_j - \mu_{j+1} < e, \forall j \geq 1$ or if μ' is e – regular ".

Mullineux conjectured of an existence of the type of the main partition but Fayers in [3] introduced the necessary conditions for this partition adding another type to it and called it the two partitions of Mullineux. We stand in this section with the Fayers conditions :

Definition(2.4) : [3]

Suppose μ is an e – regular partition, and take an abacus display for μ; (diag.(A)), with b beads, for some $b \geq \mu'_1$. let α , γ be the positions of the last bead and the first empty space on the abacus, respectively; so α is the beta number $\beta_1 = \mu_1 + b - 1$, while γ equale $b - \mu'_1$. Assuming $\mu \neq \emptyset$, there is a unique sequence $a_1 > c_1 > \ldots > a_l > c_l$ of non – negative integers satisfying the following conditions :

1. For each $1 \leq i \leq l$,position $a_i$is occupied and position c_i is empty.

2. $a_1 =$.

3. For $1 \leq i < l$, we have :

- $a_i \equiv c_i$ (mod e), and all the positions $a_i - e, a_i - 2e, \ldots, c_i + e$ are occupied .
- all the positions $c_i - 1, c_i - 2, \ldots, a_{i+1} + 1$ are empty .

4. Either :

- $a_l \equiv c_l$ (mod e), all the positions $a_l - e, \ldots, c_l + e$ are occupied, and all the position $c_l - 1, c_l - 2, \ldots, \gamma$ are empty; or
- all the position $a_l - e, a_l - 2e, \ldots$ are occupied and $c_l = \gamma$.

μ^{Δ} is to be the partition whose abacus display is obtained by moving the beads at positions $a_1, \ldots, a_l$ to positions $c_1, \ldots, c_l$.

There exist another partition; " a conjugate" definition to (2.4); as the following :

Definition (2.5) : [3]

Suppose λ is an e – restricted partition, and take an abacus display; (diag(A)), for λ with b beads . let δ, ε be the positions of the last bead and the first empty space on the abacus, respectively; Assuming $\lambda \neq \emptyset$, there is a unique sequence$f_1 > g_1 > \cdots >$

$f_u > c_u$of non – negative integers satisfying the following conditions :

1. For each $1 \leq i \leq u$, position f_i is occupied and position $g_i$is empty.

2. $g_u = \varepsilon$.

3. For $1 < i \leq u$, we have :

• $f_i \equiv g_i$ (mod e), and all the positions $f_i - e, f_i - 2e, \ldots, g_i + e$ are empty;

• all the positions $f_i + 1, f_i + 2, \ldots, g_{i-1} - 1$ are occupied .

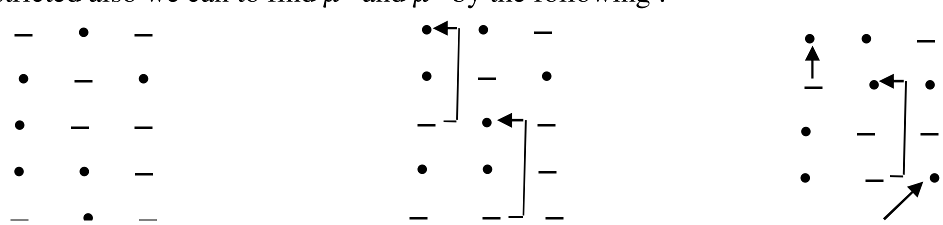
4. Either :

• $f_u \equiv g_u$ (mod e), all the positions $f_1 - e, \ldots, g_1 + e$ are empty, and all the position $\delta,\ \delta - 1, \ldots, f_1 + 1$ are occupied; or

• all the position $g_1 + e,\ g_1 + 2e, \ldots$ are empty and $f_1 = \delta$.

λ^{∇} is to be the partition whose abacus display is obtained by moving the beads at positions $f_1, \ldots, f_u$ to positions $g_1, \ldots, g_u$.

For example, let $\mu = (7, 5, 5, 3, 3, 2, 1)$ and b=7 then it's 3– regular and 3 – restricted also we can to find μ^{Δ} and μ^{∇} by the following :



μ = (7,5,5 ,3,3 ,2,1) μ^{Δ}= (4,4,3,2,1) μ^{∇} = (5,4,2,2,2)

Mahmood and Ghanem in [8] and [9] found the suitable partition for the equivalence of μ^{Δ} and μ^{∇} observe that any wrong move of any bead in μ^{Δ} or μ^{∇} may be lead to another partition. Here we try to find a new route of the value partition μ^{Δ}, which may usual in the following section .

3 e – Rim

Ford in [4] used the e – rim in Mullineux map after to find any rim in each $e \geq 2$. Through our work, we observe the value of the partition μ^{Δ} is equal to the value of a partition μ^{I} which introduced by Bessenrodt et al. in [1] .

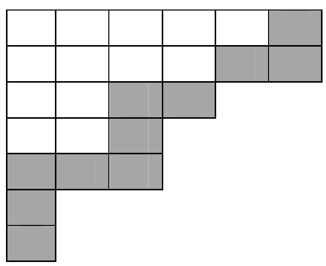
<u>Definition (3.1)</u> : [4]

Let be $e \geq 2$. The e – rim is defined by :

"Beginning at the top right – hand corner of [μ], the first e nodes of the rim are in the e – rim. Then skip to the next row, and take the next e nodes of the rim. Continue the same work about the last rows in rim; the last of these "e – Segments" my contain fewer than e nodes ".

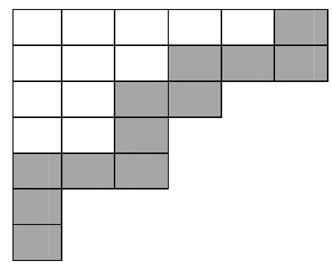
For example, if $= (6,6,4,3,3,1,1)$, then

$[\mu\ ]_{e=3}$



and

$[\mu\ ]_{e=6}$



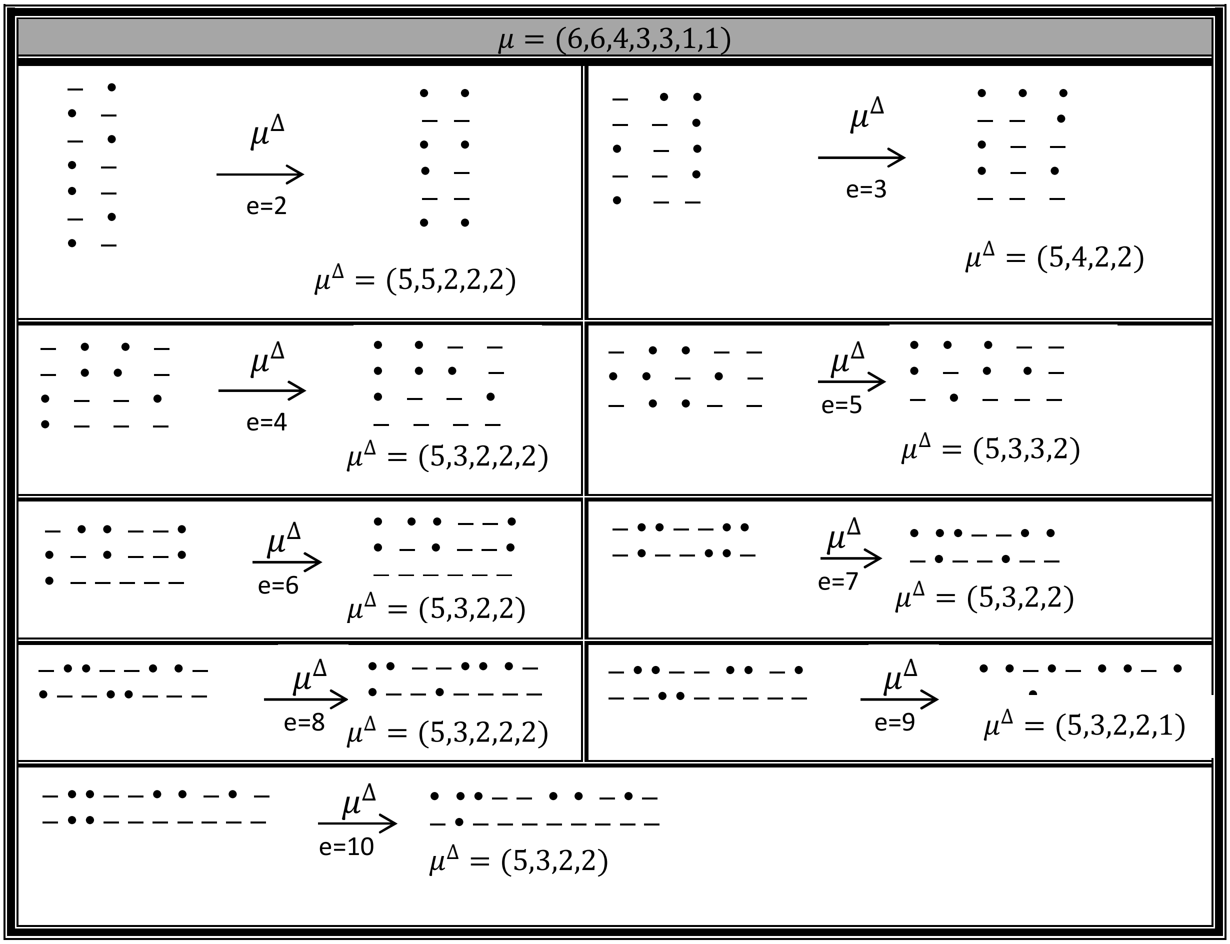
<u>Definition (3.2)</u> : [1]

Removing the e – rim from Young diagram [μ], we obtain a new e – regular partition denoted by μ^{I} .

From the above example, if $\mu = (6,6,4,3,3,1,1)$, then

$\mu = (6,6,4,3,3,1,1)$	
e	The value of μ^{I}
2	(5,5,2,2,2)
3	(5,4,2,2)
4	(5,3,2,2,2)
5	(5,3,3,2)
6	(5,3,2,2)
7	(5,3,2,2)
8	(5,3,2,2,2)
9	(5,3,2,2,1)
≥ 10	(5,3,2,2)

By chance if we used the definition of μ^{Δ} for any $e \geq 2$, then we can find the value of the partition μ^{Δ} is equal to the value of μ^{I}, from the above example :



<u>Theorem (3.3)</u> : For any partition μ of r, the partition of μ^{I} is equal to the partition of μ^{Δ}.

Proof :

By using the definition of μ^{Δ}, $a_i \equiv c_i$ (mod e), then $a_l - c_l = me$ for a positive integer m and $1 \leq i < l$. Since a_i is any part of μ; such μ_s, then μ_s will appear in the partition of μ^{Δ} as (μ_s – (me – x)) where x is the number of beads between a_i and c_i , and any part is meeting one of the bead represented in μ^{Δ} by $(\mu_{s+v} - 1)$ where $1 \leq v \leq x$.

Similarly, where $a_l \equiv c_l$ (mod e). Now, if $c_l = V$ then this part is equal to zero in μ^{Δ} Continue the same work, and we find the segments correspond the bead a_i and the beads between them and c_i is present in μ^{Δ} by delete me and this equal to the

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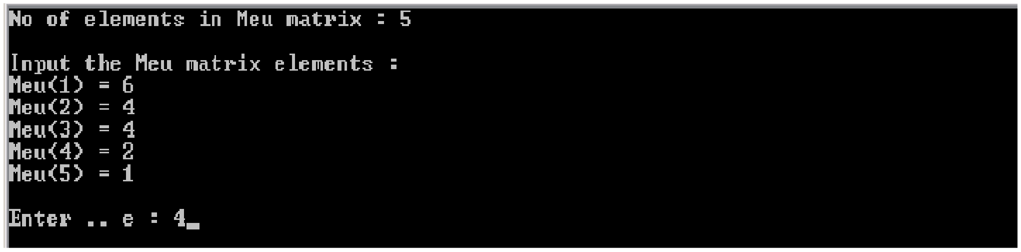
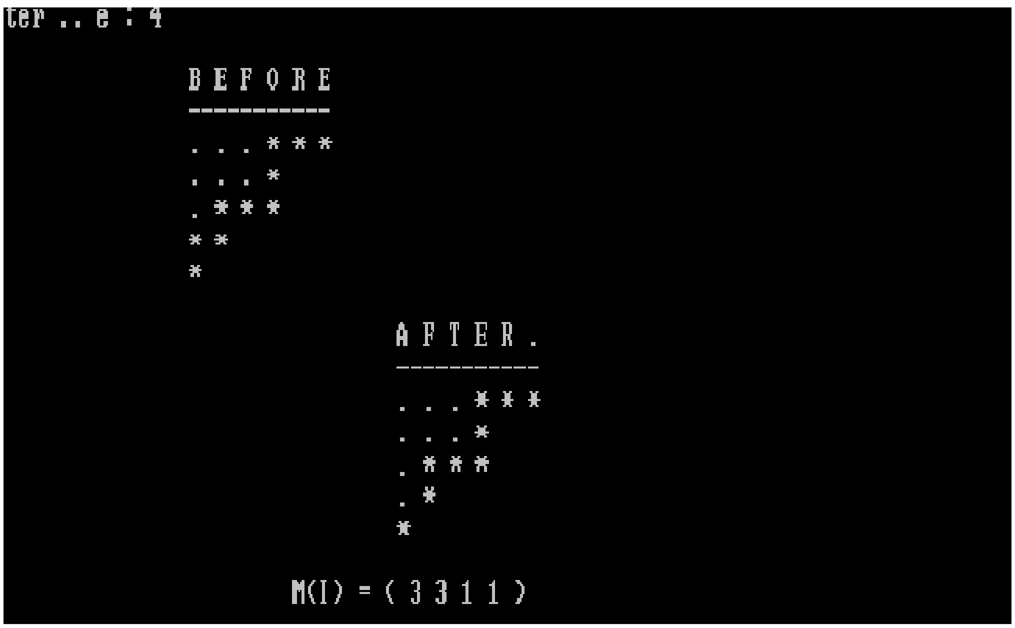
same work in μ^{I} by m – (e – segment). then $\mu^{\mathrm{I}} = \mu^{\Delta}$

We try to support the theorem (3.3) by introduced many algorithms, algorithm (1) is to find the value of the partition of μ^{I} for any $e \geq 2$, and algorithm (2) by the aid of check function is to find the diagram (A) for any μ of r by using (BEFORE) and to find μ^{Δ} by using (AFTER) Finally, we can find $\mu^{\mathrm{I}} = \mu^{\Delta}$.

Algorithm (1):

1- Start .
2- Input no. of partitions (meu).
3- Input the elements of the partitions to the partition vector M_i, where i=1,2,3,……,meu .
4- Input the value of e .
5- Create a 2-D array X:
   no. of rows = meu
   no. of columns = M_1 .
6- Reset all elements of X .
7- Set to one every element at position (i , j) which does not belong to X at position (i+1 , j+1):
   $X_{i,j} \notin X_{i+1,j+1} \rightarrow X_{i,j} = 1$.
8- Set to one all elements of the last row of X:
   $X_{meu,j} = 1$, where j=1,2,3,……,M_1 .
9- Initialize a loop counter:
   COUNTER=1 .
10- While COUNTER<2e do .
11- Reverse the array X to a new array XX such that first column of X become the last column of XX … the last column of X become the first column of XX .
12- Convert XX to a vector YY.
13- Exchange a no. of successive ones that equals to e in YY with (1,2,3,……,e) by using a counter.
14- Reconvert the vector YY to a 2-D array XX .
15- Reverse XX to the original array X .
16- Search for every element equals to value e, then reset all elements preceding it by using a reset counter and using a special counter (K) for every search process .
17- If K=COUNTER then exit loop to step 20 .
18- Increment COUNTER .
19- For every row, find and print no. of spaces which represent the values of partition .
20- End .





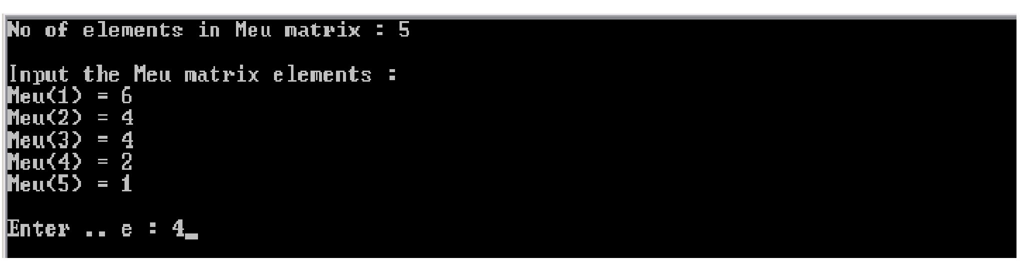
Algorithm (2):

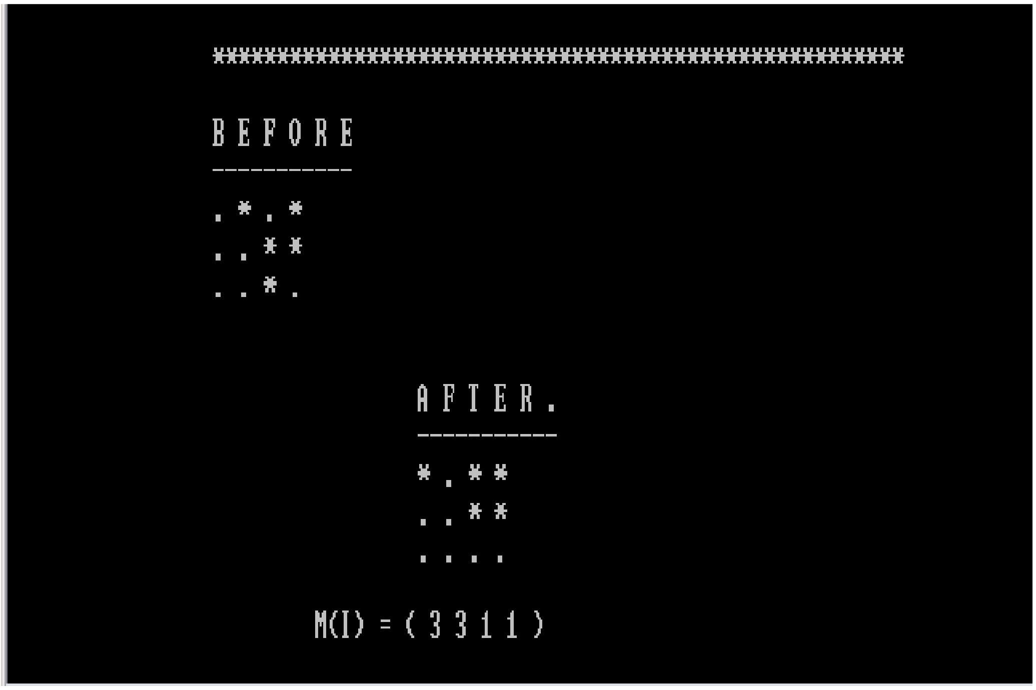
1- Start .
2- Input no. of partitions (meu).
3- Input the elements of the partitions to the partition vector M_i , where i=1, 2, 3, ……, meu .
4- Input the value of e .
5- Create the vector B and calculate for every element of it by using the formula: $B_i = M_i + meu - i$, where i=1,2,3,……,meu .
6- Find the maximum element of B known as Big .
7- Create a 2-D array Y, and find the total no. of it's elements (size), where Y's elements will be used for representing the nodes and spaces, such that: $mm = e * int(Big/(e*1.0)+1)$.

8- Find the dimensions of Y:
no. of rows = int(mm/e).
no. of columns = e .
9- Create and reset a vector YY in size equal to mm .
10- Set to one every element of YY which it's index equals to value of B_i :
$YY_{B_i} = 1$.
11- Convert YY to a 2-D array Y.
12- Search in Y for the last element equals to one, then assign it's position in (r1 , c1) .
13- Call for the search function (Check) and run it on r1, c1.
14- While ($r2 - 1 \geq 0$) do
where r2 represent the row that the Check function stops at .
15- If the Check function stops at element in the first column and it's value=0, then assign the position of the last column in the previous row
r2 = r2 -1
c2 = e -1 .
16- If the Check function stops at element equals to 0 (or element selected from step 15) and not lays in first column or first row, then search for element equals to 1 in the same row beginning from the concerning element down to first column.
17- If Check function stops not into first row but c1 value is negative, then reassign:
r2 = r2 -1
c2 = e -1
and start looking for element equals to 1 in the same row down to the first column .
18- If Check function reach the first element of Y, then reset the new selected position:
r2 = 0
c2 = 0
then exit loop to step 22 .
19- If the exchanging parameter (flag = 0) which means there is no other spaces above this element, then reset the element that the Check function stops at, and set the first element of Y, and exit the loop to step 22 .
20- Call the Check function again and run it on r2, c2.
21- Repeat loop (go to step 14) .
22- If the Check function stops at element equals to 1 and not lays in the first column, then reset this element and set tot 1 the first element in Y, and reset r2 = 0, c2 = 0 .
23- Convert Y to a vector YY.
24- For every element in YY equals to 1, count and print the preceding elements equal to 0 .
25- End .

Check function:

1- Start .
2- Receive a 2-D array and the position of the element equal to 1 and return the new position and the exchange parameter :
   Check(Y[][] , r1 , c1 , r2 , c2 , flag) .
   where (Y[][]) is the 2-D array.
   (r1 , c1) are the position of element = 1
   (r2 , c2) are the new position
   (flag) is the exchange parameter r.
3- Let (flag = 0) (meaning no exchange happened) .
4- In column c1, start a loop for it's all elements (rows) starting from r1 – 1 down to first row .
5- Search for element = 0 .
6- Reset the starting search element:
   $Y_{r1,c1} = 0$.
7- Set to one the element that has been found in searching process:
   $Y_{i,c1} = 1$
8- Select new row and select the column precedes c1:
   r2 = i
   c2 = c1 – 1 .
9- Set flag = 1 (meaning an exchange happened) .
10- Return r2 , c2 , flag and exit .
11- End .





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