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Structure of Lie Derivations on the Algebra of Locally Measurable Operators Affiliated with a von Neumann Algebra

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Abstract

Given a von Neumann algebra M denote by $LS(M)$ the algebras of all locally measurable operators affiliated with M. We prove that every Lie derivation on $LS(M)$, is in standard form, that is, it can be uniquely decomposed into the sum of a derivation and a center-valued trace.

Keywords: von Neumann algebras, measurable operator, locally measurable operator type I von Neumann algebras, derivation, inner derivation, Lie derivation, center-valued trace

1.Introduction

The structure of Lie derivations on C^*-algebras and on more general Banach algebras has attracted some attention over the past years. Let A be an algebra over the complex number. A linear operator $D : A \to A$ is called a *derivation* if $D(xy) = D(x)y + xD(y)$ for all $x, y \in A$ (Leibniz rule). Each element $a \in A$ defines a derivation D_a on A given as $D_a(x) = ax - xa$, $x \in A$. Such derivations D_a are said to be *inner derivations*. If the element a implementing the derivation D_a on A, belongs to a larger algebra B, containing A (as a proper ideal as usual) then D_a is called a *spatial derivation*. A linear operator $L : A \to A$ is called a *Lie derivation* if $L([x, y]) = [L(x), y] + [x, L(y)]$, $\forall x, y \in A$, where $[x, y] = xy - yx$. Denote by $Z(A)$ the center of A. A linear operator $\tau : A \to Z(A)$ is called *a center-valued trace* if $\tau(xy) = \tau(yx)$, $\forall x, y \in A$. The problem of the standard decomposition for a Lie derivation in rings theory was studied in work by W. S. Martindale [4]. W. S. Martindale solved this problem for primitives rings containing nontrivial idempotents and with the characteristic unequal to 2. Following these results obtained for rings,

C. Robert Miers in [5] solved the problem of the standard decomposition for the case of von Neumann algebras. The present work is devoted to the standard decomposition of Lie derivations on the algebra of locally measurable operators affiliated with a von Neumann algebra.

2.Preliminaries

Let H be a Hilbert space, $B(H)$ be the algebra of all bounded linear operators acting in H, M be a von Neumann subalgebra in $B(H)$, $P(M)$ be a complete lattice of all orthoprojections in M.

A linear subspace $\mathscr{D}$ on H is said to be *affiliated* with M (denoted as $\mathscr{D}\eta M$), if $u(\mathscr{D}) \subseteq \mathscr{D}$ for every unitary operator u from the commutant $M' = \{y \in B(H) : xy = yx, \forall x \in M\}$ of the algebra M.

A linear operator x on H with the domain $\mathscr{D}(x)$ is said to be *affiliated* with M (denoted as $x\eta M$), if $u\left(\mathscr{D}\left(x\right)\right) \subseteq \mathscr{D}(x)$ and $ux(\xi) = xu(\xi)$ for every unitary operator $u \in M'$, and all $\xi \in \mathscr{D}(x)$.

A linear subspace $\mathscr{D}$ in H is said to be *strongly dens* in H with respect to the von Neumann algebra M, if
1) $\mathscr{D}\eta M$,
2) there exists a sequence of projections $\{p_n\}_{n=1}^{\infty} \subset P(M)$, such that $p_n \uparrow \mathbf{1}$, $p_n(H) \subset \mathscr{D}$, and $p_n^{\perp} = \mathbf{1} - p_n$ is finite in M for all $n \in \mathbb{N}$, where $\mathbf{1}$ is the identity M.

A closed linear operator x, on a H, is said to be *measurable* with respect to the von Neumann algebra M, if $x\eta M$, and $\mathscr{D}(x)$ is strongly dens in H. Denote by $S(M)$ the set of all measurable operators affiliated with M.

A closed linear operator x, on a H, is said to be *locally measurable* with respect to the von Neumann algebra M, if $x\eta M$, and there exists a sequence $\{z_n\}_{n=1}^{\infty}$ of central projections in M such that $z_n \uparrow \mathbf{1}$ and $z_n x \in S(M)$ for all $n \in N$.

Denote by $LS(M)$ of all locally measurable operators with respect to M. Denote by $Z(LS(M))$ the center of $LS(M)$.

A vonNeumann algebra M is of type I if it contains a faithful abelian projection e (i.e. eMe is an abelian (commutative) von Neumann algebra).

3.Structure of Lie derivations

Let M be a von Neumann algebra with the center Z. Let L: $LS(M) \to LS(M)$ be arbitrary Lie derivation.

If p_i, p_j are projectors in $LS(M)$, then $p_i LS(M) p_j = \{p_i A p_j : A \in LS(M)\}$, $i, j = 1, 2$. Set $p_1 = p$ and $p_2 = 1 - p$. Then $LS(M) = \sum\limits_{i=1}^{2}\sum\limits_{j=1}^{2} p_i LS(M) p_j$. Let further $M_{ij} = p_i LS(M) p_j$, $i, j = 1, 2$. Recall that $M_{ij} = M_{ik} M_{kj}$, for $i, j = 1, 2$.

Lemma 1. *Let p be a projector in $LS(M)$. Then for all $x \in LS(M)$,*

$$x\left\{pL\left(p\right)+L\left(p\right)p+pL\left(p\right)p-L\left(p\right)\right\}-$$
$$-\left\{pL\left(p\right)+L\left(p\right)p+pL\left(p\right)p-L\left(p\right)\right\}x=3px\left\{pL\left(p\right)+L\left(p\right)p-L\left(p\right)\right\}-$$
$$-3\left\{pL\left(p\right)+L\left(p\right)p-L\left(p\right)\right\}xp. \tag{1}$$

Proof. The equality

$$\left[\left[\left[x,p\right],p\right],p\right]=\left[x,p\right] \tag{2}$$

holds for any $x \in LS(M)$. Applying L to the identity (2), we obtain

$$L[[[x,p],p],p]=L[x,p],$$

$$[[[Lx,p]+[x,Lp],p]+[x,Lp],p]+[[[x,p],Lp],p]+[[[x,p],p],L(p)]=[L(x),p]+[x,L(p)].$$

what implies the required equality. □

Lemma 2. $L(p)=[p,s]+z$ *for some* $s \in LS(M)$ *and* $z \in Z(LS(M))$.

Proof. Let $L(p)=\sum f_{ij}$, $f_{ij} \in M_{ij}$ $(i,j=1,2)$. Applying (1) for all $x \in S(M)$, we obtain

$$x(2f_{11}-f_{22})-(2f_{11}-f_{22})x=3px(f_{11}-f_{22})-3(f_{11}-f_{22})xp. \tag{3}$$

Let $x \in M_{12}$, then (3) implies $f_{11}x=xf_{22}$, what follows

$$(f_{11}+f_{22})x=x(f_{11}+f_{22}) \qquad (x \in M_{12}),$$

since $f_{22}x=xf_{11}=0$. Analogously, $(f_{11}+f_{22})x=x(f_{11}+f_{22})$ $(x \in M_{21})$. Let now $x \in M_{11}$ and $y \in M_{12}$. Then

$$\left\{(f_{11}+f_{22})x-x(f_{11}+f_{22})\right\}y=(f_{11}+f_{22})xy-xy(f_{11}+f_{22})=$$
$$(f_{11}+f_{22})xy-(f_{11}+f_{22})xy=0,$$

since $y, xy \in M_{12}$. It follows that

$$\left\{(f_{11}+f_{22})x-x(f_{11}+f_{22})\right\}y=0$$

for all $y \in M_{12}$. We obtain from here

$$(f_{11}+f_{22})x-x(f_{11}+f_{22})=0 \qquad (x \in M_{11}).$$

Similarly,

$$(f_{11}+f_{22})x-x(f_{11}+f_{22})=0 \qquad (x \in M_{22}),$$

i.e. $f_{11}+f_{22}=z \in Z(S(M))$. Hence, $L(p)=(f_{12}+f_{21})+z$ and, setting $s=f_{12}-f_{21}$, we obtain $L(p)=(ps-sp)+z$. □

It follows from here that it is sufficient to consider the case of $L(p) \in Z(LS(M))$, since if we prove the base theorem for this case, then, taking $L' = L - D_s$, where D_s is an inner derivation, we shall obtain the standard decomposition in the general case.

Lemma 3. *$L\left(M_{ij}\right) \subset M_{ij}$ for $i \neq j$.*

Proof. Let $x \in M_{12}$ and $L(x) = \sum y_{ij}$, $y_{ij} \in M_{ij}$ $(i, j = 1, 2)$. Then, taking into account the equality $x = [p, x]$, we obtain

$$\sum y_{ij} = L(x) = L([p, x]) = [L(p), x] + [p, L(x)] = [p, L(x)] = y_{12} - y_{21},$$

since $L(p) \in Z(LS(M))$, it follows that $y_{11} = y_{21} = y_{22} = 0$. Thus, $L(x) \in M_{12}$. The case of $x \in M_{21}$ can be proved analogously. □

Lemma 4. *$L(M_{ii}) \subset M_{ii} + Z(LS(M))$.*

Proof. Let $x \in M_{11}$ and $L(x) = \sum y_{ij}$, $y_{ij} \in M_{ij}$. Then $[p, x] = 0$ and $0 = L\left([p, x]\right) = [L(p), x] + [p, L(x)] = y_{12} - y_{21}$, so that $y_{12} = y_{21} = 0$ and $L(x) \in M_{11} + M_{22}$. Similarly, if $x \in M_{22}$, then $L(x) \in M_{11} + M_{22}$. Let $x \in M_{11}$ and $y \in M_{22}$, $L(x) = a_{11} + a_{22}$, and $L(y) = b_{11} + b_{22}$ $(a_{ii}, b_{ii} \in M_{ii})$. Then $0 = L([x, y]) = [L(x), y] + [x, L(y)] = [a_{22}, y] + [x, b_{11}] = 0$, where $[a_{22}, y] \in M_{22}$ and $[x, b_{11}] \in M_{11}$. Hence, in particular, $[a_{22}, y] = 0$ for all $y \in M_{22}$, i.e. a_{22} is a central element in M_{22}, so that $a_{22} = (\mathbf{1} - p)z$, $z \in Z(S(M))$. Therefore

$$L(x) = a_{11} + (\mathbf{1} - p)z = [(a_{11} - pz) + z] \in M_{11} + Z(LS(M)),$$

where $z \in Z(LS(M))$.

A similar argument holds if $x \in M_{22}$. □

Obtained results imply that:

if $x \in M_{ij}$ $(i \neq j)$, then $L(x) = x^* \in M_{ij}$;

if $x \in M_{ii}$, then $L(x) = x^* + z$, $x^* \in M_{ii}$, $z \in Z(LS(M))$.

One can define from these relations the mapping D from $LS(M)$ into $LS(M)$, assuming

$$D(x) = x^*, \text{ if } x \in M_{ij}, \text{ for all } i, j.$$

Now consider the mapping τ from $LS(M)$ into $Z(LS(M))$ defined by the equality

$$\tau(x) = L(x) - D(x), \quad x \in LS(M).$$

Lemma 5. *$\tau(x + y) = \tau(x) + \tau(y)$ for any $x, y \in LS(M)$.*

Proof. It is sufficient to prove additivity of τ on M_{ii}. If $x, y \in M_{ii}$, then $\tau(x + y) - \tau(x) - \tau(y) = L(x + y) - D(x + y) - L(x) + D(x) - L(y) + D(y) = [D(x) + D(y) - D(x + y)] \in M_{ii} \cap Z(LS(M)) = 0$. □

Corollary. *$D(x+y) = D(x) + D(y)$ for any $x, y \in LS(M)$.*

Lemma 6. *$D(xyx) = D(x)yx + xD(y)x + xyD(x)$ for all $x \in M_{ij}$ $(i \neq j)$ and for all $y \in LS(M)$.*

Proof. $2xyx = [[x, y], x]$ for $x \in M_{ij}$ $(i \neq j)$. Then

$$2D(xyx) = L(2xyx) = L([[x, y]x]) = [[L(x), y] + [x, L(y)], x] + [[x, y], L(x)] =$$

$$= [[D(x), y] + [x, D(y)], x] + [[x, y], D(x)] = 2\{D(x)yx + xD(y)x + xyD(x)\}.$$

□

Lemma 7. *$D(xy) = D(x)y + xD(y)$ for $x \in M_{ii}$ and $y \in M_{jk}$ $(j \neq k)$.*

Proof. Let $x \in M_{11}$ and $y \in M_{12}$. Then $D(xy) = L(xy) - \tau(xy) = L(xy), \tau(xy) = 0$. Therefore $D(xy) = L[x, y] = [L(x), y] + [x, L(y)] = [(D+\tau)(x), y] + [x, (D+\tau)(y)] = [D(x), y] + [x, D(y)]$, because $\tau(x), \tau(y)$ is central elements, hence $[\tau(x), y] = [x, \tau(y)] = 0$. It follows that $D(xy) = D(x)y + xD(y)$, since $yD(x) = D(y)x = 0$. The case of $x \in M_{22}, y \in M_{21}$ can be proved analogously. □

Lemma 8. *$D(xy) = D(x)y + xD(y)$ for $x \in M_{ii}$ and $y \in M_{jj}$.*

Proof. Let $x, y \in M_{11}$, $r \in M_{12}$, then by lemma 7 we have

$$D((xy)r) = D(xy)r + xyD(r)$$

.

$$D(xy)r = D(xyr) - xyD(r) = D(x)yr + xD(yr) - xyD(r) = D(x)yr+$$

$$+x\{D(y)r + yD(r)\} - xyD(r) = \{D(x)y + xD(y)\}r.$$

Hence, $\{D(xy) - D(x)y - xD(y)\}r = 0$ for all $r \in M_{12}$. It follows that $D(xy) - D(x)y - xD(y) = 0$. The case of $x \in M_{22}, y \in M_{22}$ can be proved analogously. □

Theorem 1. *D is a derivation from $LS(M)$ into $LS(M)$.*

Proof. Let $x \in M_{12}$ and $y \in M_{21}$. We have

$$\tau([x, y]) = L([x, y]) - D([x, y]) = [L(x), y] + [x, L(y)] - D([x, y]) =$$

$$= [D(x), y] + [x, D(y)] - D(xy) + D(yx),$$

what imply

$$\{D(x)y + xD(y) - D(xy)\} + \{D(yx) - D(y)x - yD(x)\} = z \in Z(LS(M)). \quad (4)$$

If $z = 0$, then $[D(x)y + xD(y) - D(xy)] \in M_{11} \cap M_{22}$, i.e. $D(x)y + xD(y) - D(xy) = 0$. Suppose $z \neq 0$. Multiplying the equality (4) on x from the left, we obtain $xD(yx) - xD(y)x - xyD(x) = xz$. Applying Lemma 7, we have $D(xyx) - D(x)yx - xD(y)x - xyD(x) = xz$. According to Lemma 6 we obtain $xz = 0$, what follows $x = 0$ and we obtain the required equality. □

Corollary. $\tau(xy - yx) = 0$ *for all* $x, y \in LS(M)$.

Thus, as the final result, we obtain the following main theorem.

Theorem 2. *Any Lie derivation on* $LS(M)$ *can be uniquely represented in the form of*

$$L = D + \tau$$

where D *is a derivation and* τ *is a center-valued trace from* $LS(M)$ *into* $Z(LS(M))$.

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