International Mathematical Forum, Vol. 7, 2012, no. 53, 2613 - 2616

A Note on the Density of Certain Sets of Positive Integers

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In memory of my sister Fedra Marina Jakimczuk (1970-2010)

Abstract

In former articles we have obtained a set A of positive integers with positive density σ and a partition of A in infinite sets A_i $(i = 1, 2, \ldots)$ with positive density σ_i such that the following equation holds $\sum_{i=1}^{\infty} \sigma_i = \sigma$. Consequently the sum of the densities of the infinite sets A_i equals the density of the union A of these infinite sets. In this note we give examples where the following inequality holds $\sum_{i=1}^{\infty} \sigma_i < \sigma$. That is, the sum of the densities of the infinite sets A_i is less than the density of the union A of these infinite sets.

Mathematics Subject Classification: 11A99, 11B99

Keywords: Sets of positive integers, density

1 Introduction

In this section p_n denotes the n-th prime number. Then $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11, \ldots$.

If A is a set of infinite positive integers and $A(x)$ is the number of positive integers in A that do not exceed x the density of the set A is

$$\lim_{x \to \infty} \frac{A(x)}{x},$$

when this limit exists. Clearly the density is a nonnegative real number less than or equal to 1.

Consider the following two examples.

Example 1.1 *Let β_{p_h} be the set of all positive integers whose prime factorization is of the form $p_h^{s_h} p_{h+1}^{s_{h+1}} \ldots$ where $s_i \geq 0 \quad (i = h+1, h+2, \ldots)$ and $s_h \geq 1$. That is, the set β_{p_h} of all positive integers such that the minimum prime factor in their prime factorization is p_h. Note that if $i \neq j$ then the sets β_{p_i} and β_{p_j} are disjoints. On the other hand $\bigcup_{i=1}^{\infty} \beta_{p_i} = N - \{1\}$, where N is the set of all positive integers. That is, the sets β_{p_h} $(h = 1, 2, \ldots)$ are a partition of $N - \{1\}$. The density of $N - \{1\}$ is 1. In [2] is proved that the set β_{p_h} $(h = 1, 2, \ldots)$ has positive density*

$$D_{p_h} = \left(\prod_{i=1}^{h-1} \left(1 - \frac{1}{p_i} \right) \right) \frac{1}{p_h},$$

and that the sum of the infinite positive densities is 1. That is,

$$\sum_{h=1}^{\infty} D_{p_h} = 1.$$

Consequently the sum of the densities of the infinite sets β_{p_h} equals the density of the union of these infinite sets.

Example 1.2 *A positive integer n is quadratfrei if it is either a product of different primes or 1. For example, $n = 2$ and $n = 5.7.23$ are quadratfrei. Let Q_1 be the set of quadratfrei numbers, it is well-known [1, Chapter XVIII, Theorem 333] that this set has positive density $\frac{6}{\pi^2}$. That is, if $Q_1(x)$ is the number of quadratfrei numbers not exceeding x we have*

$$\lim_{x \to \infty} \frac{Q_1(x)}{x} = \frac{6}{\pi^2}.$$

Let Q_2 be the set of not quadratfrei numbers. That is, the set of numbers such that in their prime factorization there exists a prime with exponent greater than 1. The density of this set will be $1 - \frac{6}{\pi^2}$. That is, if $Q_2(x)$ is the number of not quadratfrei numbers not exceeding x we have

$$\lim_{x \to \infty} \frac{Q_2(x)}{x} = 1 - \frac{6}{\pi^2}.$$

Let us consider the set β_{p_h} of all positive integers such that in their prime factorization p_h is the minimum prime with exponent greater than 1. Note that if $i \neq j$ then the sets β_{p_i} and β_{p_j} are disjoint. On the other hand $\bigcup_{h=1}^{\infty} \beta_{p_h} = Q_2$. That is, the sets β_{p_h} $(h = 1, 2, \ldots)$ are a partition of Q_2. The density of Q_2 is (see above)$1 - \frac{6}{\pi^2}$. In [3] is proved that the set β_{p_h} $(h = 1, 2, \ldots)$ has positive density

$$D_{p_h} = \left(\prod_{i=1}^{h-1} \left(1 - \frac{1}{p_i^2} \right) \right) \frac{1}{p_h^2}.$$

and that the sum of the infinite positive densities is $1 - \frac{6}{\pi^2}$. *That is,*

$$\sum_{h=1}^{\infty} D_{p_h} = 1 - \frac{6}{\pi^2}.$$

Consequently the sum of the densities of the infinite sets β_{p_h} *equals the density of the union of these infinite sets.*

In these two examples we have a set A of positive integers with positive density σ and a partition of A in infinite sets A_i $(i = 1, 2, \ldots)$ with positive density σ_i such that the following equation holds

$$\sum_{i=1}^{\infty} \sigma_i = \sigma. \tag{1}$$

Consequently the sum of the densities of the infinite sets A_i equals the density of the union A of these infinite sets.

2 Main Results

We have the following theorem.

Theorem 2.1 *Let A be a set of positive integers with density σ and consider a partition of A in infinite sets A_i $(i = 1, 2, \ldots)$ with density σ_i. The following inequality holds*

$$\sum_{i=1}^{\infty} \sigma_i \leq \sigma. \tag{2}$$

Proof. We have for all $k \geq 1$ the inequality

$$A_1(x) + A_2(x) + \cdots + A_k(x) \leq A(x) \qquad (x \geq 1),$$

where $A_i(x)$ is the number of numbers in the set A_i that do not exceed x and $A(x)$ is the number of numbers in the set A that do not exceed x.

Consequently

$$\lim_{x\to\infty} \left(\frac{A_1(x) + A_2(x) + \cdots + A_k(x)}{x} \right) = \sigma_1 + \sigma_2 + \cdots + \sigma_k \leq \lim_{x\to\infty} \frac{A(x)}{x} = \sigma$$

Therefore since the $\sigma_i \geq 0$ the series $\sum_{i=1}^{\infty} \sigma_i$ is convergent and $\sum_{i=1}^{\infty} \sigma_i \leq \sigma$. The theorem is proved.

In Examples 1.1 and 1.2 equation (2) becomes the equality

$$\sum_{i=1}^{\infty} \sigma_i = \sigma.$$

However there exist examples where

$$\sum_{i=1}^{\infty} \sigma_i < \sigma. \tag{3}$$

We now give two examples where (3) is fulfilled.

Example 2.2 *Let π_i be the set of quadratfrei with i prime factors $(i \geq 1)$ and let $\pi_i(x)$ be the number of quadratfrei with i prime factors that do not exceed x. We have [1, Chapter XXII, Theorem 437]*

$$\pi_i(x) \sim \frac{x\,(\log\log x)^{i-1}}{(i-1)!\log x} \qquad (i \geq 1).$$

Consequently

$$\sigma_i = \lim_{x\to\infty} \frac{\pi_i(x)}{x} = 0.$$

That is, the set π_i $(i \geq 1)$ has density $\sigma_i = 0$. On the other hand the union of the sets π_i $(i \geq 1)$ is the set Q_1 of quadratfrei numbers whose density is (see Example 1.2) $\sigma = \frac{6}{\pi^2}$. Consequently in this example we have

$$\sum_{i=1}^{\infty} \sigma_i = 0 < \sigma = \frac{6}{\pi^2}.$$

Example 2.3 *Let us consider the set (see Example 1.2 and Example 2.2) $A_i = \pi_i \cup \beta_{p_i}$ $(i \geq 1)$. This set has positive density $\sigma_i = 0 + D_{p_i} = D_{p_i}$. On the other hand, the union of the sets A_i $(i \geq 1)$ is the set of all positive integers whose density is $\sigma = 1$. Consequently in this example we have*

$$\sum_{i=1}^{\infty} \sigma_i = \sum_{i=1}^{\infty} D_{p_i} = 1 - \frac{6}{\pi^2} < \sigma = 1.$$

ACKNOWLEDGEMENTS. The author is very grateful to Universidad Nacional de Luján.

References

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Received: June, 2012