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On Generalized k-Primary Rings

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Abstract. The present paper introduces and studies some new types of rings and ideals such as generalized k-primary rings (resp. generalized k-primary ideals), principally generalized k-primary rings (resp. principally generalized k-primary ideals) and completely generalized k-primary rings (resp. completely generalized k-primary ideals). Some properties of each are obtained and some characterizations of each type are given.

Keywords: $gkp-$ideal, $pgkp-$ideal, $cgkp-$ideal, gkp-ring, $pgkp$-ring, $cgkp$-ring

1 Introduction

Let R be a ring. R is called a prime ring if whenever $a, b \in R$ such that $aRb = 0$ then $a = 0$ or $b = 0$ [3] and it is called a simple ring if $R^2 \neq 0$ and contains no non trivial ideals [7] and R is called a medial ring if $abcd = acbd$ for all $a, b, c, d \in R$ [2]. If $a \in R$ then the principal ideal generated by a, denoted by $\langle a \rangle$, is defined as $\langle a \rangle = \{na + ar + sa + tau : r, s, t, u \in R, n \in Z\} = Za + aR + Ra + RaR$, and the principal right ideal of R generated by a, denoted by $\langle a \rangle_r$, is defined to be $\langle a \rangle_r = \{na+ar : n \in Z \text{ and } r \in R\}$=$Za+aR$ and the principal left ideal of R generated by a, denoted by $\langle a \rangle_l$, is defined as $\langle a \rangle_l = \{na + ra : n \in Z \text{ and } r \in R\}$=$Za + Ra$ [1], where $Za = \{na : n \in Z\}$, then clearly $\langle a \rangle_r \subseteq \langle a \rangle$ and $\langle a \rangle_l \subseteq \langle a \rangle$. R is called a Noetherian (resp. an Artinian) ring if R satisfies the acc (resp. dcc) on ideals. An element $r \in R$ is called a nilpotent element if $a^n = 0$, for some positive integer n. An ideal I of R is called a nil ideal if every element of I is nilpotent and I is called a nilpotent ideal if $I^n = 0$, for some positive integer n. By saying that R is a nil ring (resp. a nilpotent ring) we mean that R is nil (resp. nilpotent) as an ideal.

2 The Results

Throughout this paper, R is a ring, need not be commutative and not necessarily containss identity, unless otherwise stated and now, we introduce the following definitions and giving some examples.

Definition2.1. Let R be a ring. We call an ideal I of R a generalized $k-$primary ideal (simply, a $gkp-$ideal) if whenever A and B are ideals of R such that $AB \subseteq I$ then $A^k \subseteq I$ or $B^k \subseteq I$ for some $k \in Z^+$ and we call R a generalized $k-$primary ring (simply, a $gkp-$ring) if the zero ideal of R is a $gkp-$ideal, equivalently R is a $gkp-$ring if A and B are ideals of R such that $AB = 0$ then A is nilpotent or B is nilpotent.

As examples, we see that the ideals $\langle 4\rangle$ and $\langle 8\rangle$ of Z are $gkp-$ideals and Z_4, Z_8 are $gkp-$rings, while the ideal $\langle 6\rangle$ of Z is not a $gkp-$ideal ($\langle 2\rangle\langle 3\rangle = \langle 6\rangle \subseteq \langle 6\rangle$ but $\langle 2\rangle^n \nsubseteq \langle 6\rangle$ and $\langle 3\rangle^n \nsubseteq \langle 6\rangle$ for all $n \in Z^+$) and the ring Z_6 is not a $gkp-$ring, since $\{0,3\}\{0,2,4\} = \{0\}$ but neither $\{0,3\}$ is nilpotent nor $\{0,2,4\}$.

Definition2.2. We call an ideal I of R a principal generalized $k-$primary ideal (simply, a $pgkp-$ideal) if whenever A and B are principal ideals of R such that $AB \subseteq I$ then $A^k \subseteq I$ or $B^k \subseteq I$ for some $k \in Z^+$ and we call R a principal generalized $k-$primary ring (simply, a $pgkp-$ring) if the zero ideal of R is a $pgkp-$ideal.

It is clear that if R is a $gkp-$ring then it is a $pgkp-$ring, since if $\langle a\rangle$ and $\langle b\rangle$ are principal ideals of R such that $\langle a\rangle\langle b\rangle = 0$ then trivially $\langle a\rangle$ is nilpotent or $\langle b\rangle$ is nilpotent while the converse is not true in general, and we give an example for this later.

Definition2.3. We call an ideal I of a ring R a completely generalized $k-$primary ideal (simply, a $cgkp-$ideal) if a and b are elements of R such that $ab \in I$ then $a^k \in I$ or $b^k \in I$ for some $k \in Z^+$ and we call R a completely generalized $k-$primary ring (simply, a $cgkp-$ring) if the zero ideal of R is a $cgkp-$ideal, equivalently if a and b are any elements of R with $ab = 0$, then a is nilpotent or b is nilpotent.

It is necessary to mention that every prime ring is a $gkp-$ring, since if R is a prime ring and A, B are ideals of R such that $AB = 0$ and if A is not nilpotent, then $A \neq 0$, so there exists $0 \neq a \in A$. Now, for all $b \in B$, we have $aRb \subseteq AB = 0$ and as R is a prime ring, we get $a = 0$ or $b = 0$ and as $a \neq 0$, we have $b = 0$, so that $B = 0$ and thus B is nilpotent. Also, it can be shown that $cgkp-$rings are independent from $pgkp-$ rings and $gkp-$rings. Now, we give the following example to establish this fact.

It is known that Z_2 is a prime ring and one can easily check that $M_{2\times 2}[Z_2]$ is also prime and thus it is a $gkp-$ring and hence a $pgkp-$ring, but it is not a $cgkp-$ring since if we take $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M_{2\times 2}[Z_2]$, then clearly $AB = 0$ but neither A is nilpotent nor B. Next, if S and T are two simple nil rings which are not nilpotent(such rings always exist, see [6]), then $R = S \bigoplus T$ is a nil ring, identifying S and T as ideals of R we have $ST = 0$. Since S and T are simple they are principal and since they are not nilpotent, R is not a $pgkp-$ring (and hence not a $gkp-$ring), but on the other hand as R is nil, each of its elements is nilpotent, so R is trivially a $cgkp-$ring.

Now, it is the time to give some characterizations of $gkp-$ideals ($gkp-$rings), $pgkp-$ideals ($pgkp-$rings) and $cgkp-$ideals ($cgkp-$rings).

The first result is the following theorem which can be proved just by applying the definitions 2.1, 2.2 and 2.3.

Theorem2.4. If I is an ideal of a ring R, then:

(1) I is a $gkp-$ideal if and only if $\frac{R}{I}$ is a $gkp-$ring.

(2) I is a $pgkp-$ideal if and only if $\frac{R}{I}$ is a $pgkp-$ring.

(3) I is a $cgkp-$ideal if and only if $\frac{R}{I}$ is a $cgkp-$ring.

Next, we prove that $gkp-$ideals can be characterized by left as well as right ideals.

Theorem2.5. For any ring R the following are equivalent.

(1) I is a $gkp-$ideal of R.

(2) If A and B are right ideals of R such that $AB \subseteq I$, then $A^k \subseteq I$ or $B^k \subseteq I$ for some $k \in Z^+$.

(3) If A and B are left ideals of R such that $AB \subseteq I$, then $A^k \subseteq I$ or $B^k \subseteq I$, for some $k \in Z^+$.

Proof. $(1 \Rightarrow 2)$ Let I be a $gkp-$ideal of R and A, B are right ideals of R such that $AB \subseteq I$. We have $\langle A\rangle RB = (A + RA)RB = ARB + RARB \subseteq AB + RAB \subseteq I$. Since $\langle A\rangle$ and RB are ideals of R and I is a $gkp-$ideal of R, we get $\langle A\rangle^{k'} \subseteq I$ or $(RB)^{k'} \subseteq I$, for some $k' \in Z^+$. If $\langle A\rangle^{k'} \subseteq I$ and as $A \subseteq \langle A\rangle$, we get $A^{k'} \subseteq I$ and if $(RB)^{k'} \subseteq I$, then as $BB \subseteq RB$, we have $B^{2k'} = (B^2)^{k'} = (BB)^{k'} \subseteq (RB)^{k'} \subseteq I$. Let $k = 2k'$, then clearly we have $A^k = A^{2k'} \subseteq A^{k'} \subseteq I$ or $B^k = B^{2k'} \subseteq I$.

$(2 \Rightarrow 1)$ Since every ideal is a right ideal the result follows at once.

$(1 \Leftrightarrow 3)$ We have $AR\langle B\rangle = AR(B+BR) = ARB+ARBR \subseteq AB+ABR \subseteq I$, then we use the same technique as in the above and getting the result.

By taking $I = 0$ in Theorem 2.5, we can give the following corollary.

Corollary2.6. For any ring R the following are equivalent:

(1) R is a $gkp-$ring.

(2) If A and B are right ideals of R such that $AB = 0$ then A is nilpotent or B is nilpotent.

(3) If A and B are left ideals of R such that $AB = 0$, then A is nilpotent or B is nilpotent.

In the following theorem we prove that the $gkp-$ideals of medial rings and medial $gkp-$rings can be characterized by a finite number of ideals.

Theorem2.7. Let R be a medial ring and I is an ideal of R, then:

(1) I is a $gkp-$ideal if and only if for any finite number of ideals $A_1, A_2, ..., A_n$ of R with $A_1A_2...A_n \subseteq I$, then $A_m^k \subseteq I$ for some $m(1 \leq m \leq n)$ and some $k \in Z^+$.

(2) R is a $gkp-$ring if and only if for any finite number of ideals of R with $A_1A_2...A_n = 0$, then at least one of the A_i,s is nilpotent.

Proof. (1) Let I be a $gkp-$ideal and $A_1, A_2, ..., A_n, (n \in Z^+)$ are ideals of R such that $A_1A_2...A_n \subseteq I$, then we have $A_1^{k'} \subseteq I$ or $(A_2...A_n)^{k'} \subseteq I$, for some $k' \in Z^+$. If the former holds then the proof will be complete and if the later holds then as R is a medial ring we have $A_2^{k'}...A_n^{k'} = (A_2...A_n)^{k'} \subseteq I$. As before we can get $A_2^{k'k''} = (A_2^{k'})^{k''} \subseteq I$ or $A_3^{k'k''}...A_n^{k'k''} = (A_3^{k'}...A_n^{k'})^{k''} \subseteq I$ for some $k'' \in Z^+$. Since we have only a finite number of ideals so by continuing this argument a finite number of steps, we get, at the end, an $m(1 \leq m \leq n)$ and some $k \in Z^+$ such that $A_m^k \subseteq I$. The proof of the converse side will follows directly from the definition of a $gkp-$ideal.

(2) Since we have R is a $gkp-$ring if and only if the zero ideal of R is a $gkp-$ideal, so by putting $I = 0$ in (1) the proof will follows at once.

Theorem2.8. Let I be an ideal of a ring R. Then the following statements are equivalent.

(1) I is a $pgkp-$ideal.

(2) If A and B are ideals of R such that $AB \subseteq I$, then for each $a \in A$ and each $b \in B$ there exists $k \in Z^+$ such that $\langle a\rangle^k \subseteq I$ or $\langle b\rangle^k \subseteq I$.

(3) If A and B are finitely generated ideals of R with $AB \subseteq I$, then there exists $k \in Z^+$ such that $A^k \subseteq I$ or $B^k \subseteq I$.

Proof. ($1 \Rightarrow 2$) Let I be a $pgkp-$ideal and $AB \subseteq I$, where A and B are ideals of R. If $a \in A$ and $b \in B$, then we have $\langle a\rangle\langle b\rangle \subseteq AB \subseteq I$ and as I is a $pgkp-$ideal we get $\langle a\rangle^k \subseteq I$ or $\langle b\rangle^k \subseteq I$ for some $k \in Z^+$.

($2 \Rightarrow 1$) Suppose $a, b \in R$ such that $\langle a\rangle\langle b\rangle \subseteq I$. As $a \in \langle a\rangle$ and $b \in \langle b\rangle$, by the hypothesis we get $\langle a\rangle^k \subseteq I$ or $\langle b\rangle^k \subseteq I$ for some $k \in Z^+$. Hence I is a $pgkp-$ideal.

($1 \Rightarrow 3$) Let I be a $pgkp-$ideal and A, B are finitely generated ideals of R with $AB \subseteq I$, then there exist $l, s \in Z^+$ such that $A = \langle a_1\rangle + \langle a_2\rangle + ... + \langle a_l\rangle$ and $B = \langle b_1\rangle + \langle b_2\rangle + ... + \langle b_s\rangle$, where $a_i, b_i \in R$. If for each $a_i \in A$ there exists $n_i \in Z^+$ such that $\langle a_i\rangle^{n_i} \subseteq I$, then by [5, Lemma 2.9] we have $A^m \subseteq I$ for some $m \in Z^+$ and if for some $a \in A$ we have $\langle a\rangle^n \nsubseteq I$ for every $n \in Z^+$, then for each $b_i \in B$ we have $\langle a\rangle\langle b_i\rangle \subseteq AB \subseteq I$ and as I is a $pgkp-$ideal so we get $\langle b_i\rangle^{s_i} \subseteq I$, for some $s_i \in Z^+$ and again by [5, Lemma 2.9] we have $B^{k'} \subseteq I$ for some $k' \in Z^+$. If we set $k =$max$\{m, k'\}$, then we get $A^k \subseteq A^m \subseteq I$ or $B^k \subseteq B^{k'} \subseteq I$ as required.

($3 \Rightarrow 1$) Suppose $a, b \in R$ such that $\langle a\rangle\langle b\rangle \subseteq I$. As $\langle a\rangle$ and $\langle b\rangle$ are finitely generated ideals of R, so by the hypothesis the result follows directly.

By taking $I = 0$ in Theorem 2.8, we can give the following corollary.

Corollary2.9. For any ring R the following are equivalent.

(1) R is a $pgkp-$ring.

(2) If A and B are ideals of R such that $AB = 0$ then for each $a \in A$ and each $b \in B$, we have either $\langle a\rangle$ is nilpotent or $\langle b\rangle$ is nilpotent.

(3) If A and B are finitely generated ideals of R with $AB = 0$, then A is nilpotent or B is nilpotent.

Before proving the next result, we mention that, if R is a ring and $a, b \in R$, then $(aRb, +)$ is a subgroup of $(R, +)$ and since $aRb = (-a)Rb = aR(-b) = (-a)R(-b)$, so one can easily check that $(Za)R(Zb) = (Za)Rb = aR(Zb) = aRb$. Now, we use this fact in the proof of the following theorem.

Theorem2.10. Let I be a $pgkp-$ideal of a ring R and $a, b \in R$. Then:

(1) If $aRb \subseteq I$, then $\langle a\rangle^k \subseteq I$ or $\langle b^2\rangle^k \subseteq I$ for some $k \in Z^+$.

(2) If $aRb \subseteq I$, then $\langle a^2\rangle^k \subseteq I$ or $\langle b\rangle^k \subseteq I$ for some $k \in Z^+$.

(3) If $\langle a\rangle_r\langle b\rangle_r \subseteq I$, then $\langle a\rangle^k \subseteq I$ or $\langle b^2\rangle^k \subseteq I$ for some $k \in Z^+$.

(4) If $\langle a\rangle_r\langle b\rangle_r \subseteq I$, then $\langle a^2\rangle^k \subseteq I$ or $\langle b\rangle^k \subseteq I$ for some $k \in Z^+$.

(5) If $\langle a\rangle_l\langle b\rangle_l \subseteq I$, then $\langle a\rangle^k \subseteq I$ or $\langle b^2\rangle^k \subseteq I$ for some $k \in Z^+$.

(6) If $\langle a\rangle_l\langle b\rangle_l \subseteq I$, then $\langle a^2\rangle^k \subseteq I$ or $\langle b\rangle^k \subseteq I$ for some $k \in Z^+$.

Proof. (1) We have $\langle a\rangle R\langle b\rangle = (Za + Ra + aR + RaR)R(Zb + Rb + bR + RbR) = aRb + aRRb + aRbR + aRRbR + RaRb + RaRRb + RaRbR + RaRRbR + aRRb + aRRRb + aRRbR + aRRRbR + RaRRb + RaRRRb + RaRRbR + RaRRRbR \subseteq aRb + aRbR + RaRb + RaRbR \subseteq I$. Since I is a $pgkp-$ideal and $\langle a\rangle, R\langle b\rangle$ are ideals of R, by Theorem 2.8, for every $x \in \langle a\rangle$ and $y \in R\langle b\rangle$, there exists $m \in Z^+$, such that $\langle x\rangle^m \subseteq I$ or $\langle y\rangle^m \subseteq I$. As $a \in \langle a\rangle$ and $b^2 = bb \in R\langle b\rangle$, we have $\langle a\rangle^k \subseteq I$ or $\langle b^2\rangle^k \subseteq I$, for some $k \in Z^+$.

(2) If we start with the ideals $\langle a\rangle R, \langle b\rangle$ then by the same technique as we have done in proving (1) we get the result.

(3) and (4). If $\langle a\rangle_r\langle b\rangle_r \subseteq I$, then we have $aRb \subseteq \langle a\rangle_r R\langle b\rangle_r \subseteq \langle a\rangle_r\langle b\rangle_r \subseteq I$ and by (1) we have $\langle a\rangle^k \subseteq I$ or $\langle b^2\rangle^k \subseteq I$ for some $k \in Z^+$ and by (2) we have $\langle a^2\rangle^k \subseteq I$ or $\langle b\rangle^k \subseteq I$ for some $k \in Z^+$.

Proceeding as in (3) and (4), we can prove (5) and (6).

Taking $I = 0$ in Theorem 2.10 we can give the following corollary.

Corollary2.11. Let R be a $pgkp-$ring and $a, b \in R$, then:

(1) If $aRb = 0$, then $\langle a\rangle$ is nilpotent or $\langle b^2\rangle$ is nilpotent.

(2) If $aRb = 0$, then $\langle a^2\rangle$ is nilpotent or $\langle b\rangle$ is nilpotent.

(3) If $\langle a\rangle_r\langle b\rangle_r = 0$, then $\langle a\rangle$ is nilpotent or $\langle b^2\rangle$ is nilpotent.

(4) If $\langle a\rangle_r\langle b\rangle_r = 0$, then $\langle a^2\rangle$ is nilpotent or $\langle b\rangle$ is nilpotent.

(5) If $\langle a\rangle_l\langle b\rangle_l = 0$, then $\langle a\rangle$ is nilpotent or $\langle b^2\rangle$ is nilpotent.

(6) If $\langle a\rangle_l\langle b\rangle_l = 0$, then $\langle a^2\rangle$ is nilpotent or $\langle b\rangle$ is nilpotent.

Theorem2.12. Let R be a ring and $a, b \in R$ and I is an ideal of R. If R satisfies the condition $\langle x\rangle^2 = \langle x^2\rangle$ for every $x \in R$, then the following statements are equivalent.

(1) I is a $pgkp-$ideal.

(2) If $aRb \subseteq I$, then $\langle a\rangle^k \subseteq I$ or $\langle b\rangle^k \subseteq I$ for some $k \in Z^+$.

(3) If $\langle a\rangle_r \langle b\rangle_r \subseteq I$, then $\langle a\rangle^k \subseteq I$ or $\langle b\rangle^k \subseteq I$ for some $k \in Z^+$.

(4) If $\langle a\rangle_l \langle b\rangle_l \subseteq I$, then $\langle a\rangle^k \subseteq I$ or $\langle b\rangle^k \subseteq I$ for some $k \in Z^+$.

proof. $(1 \Rightarrow 2)$ Suppose I is a *pgkp*$-$ideal and let $aRb \subseteq I$, then by Theorem 2.10, we have $\langle a\rangle^m \subseteq I$ or $\langle b^2\rangle^m \subseteq I$ for some $m \in Z^+$. The later case gives $\langle b\rangle^{2m} \subseteq I$ and by taking $k = 2m$, we get $\langle a\rangle^k \subseteq \langle a\rangle^m \subseteq I$ or $\langle b\rangle^k \subseteq I$.

$(2 \Rightarrow 1)$ Suppose that (2) holds and for $a, b \in R$ we have $\langle a\rangle\langle b\rangle \subseteq I$, then we get $aRb \subseteq \langle a\rangle\langle b\rangle \subseteq I$ and so by the given hypothesis there exists $k \in Z^+$ such that $\langle a\rangle^k \subseteq I$ or $\langle b\rangle^k \subseteq I$ and thus I is a *pgkp*$-$ideal.

$(1 \Rightarrow 3)$ By using the same argument as in $(1 \Rightarrow 2)$, we get the result.

$(3 \Rightarrow 1)$ By using the fact that $\langle a\rangle_r\langle b\rangle_r \subseteq \langle a\rangle\langle b\rangle$, then proceeding as in the proof of $(2 \Rightarrow 1)$ the result will follows directly.

$(1 \Leftrightarrow 4)$ In a similar argument to the proof of $(1 \Leftrightarrow 3)$, we can prove this part.

Putting $I = 0$ in Theorem 2.12, we get the following corollary.

Corollary2.13. Let R be a ring and $a, b \in R$. If R satisfies the condition $\langle x\rangle^2 = \langle x^2\rangle$ for every $x \in R$, then the following statements are equivalent.

(1) R is a *pgkp*$-$ring.

(2) If $aRb = 0$, then $\langle a\rangle$ is nilpotent or $\langle b\rangle$ is nilpotent.

(3) If $\langle a\rangle_r\langle b\rangle_r = 0$, then $\langle a\rangle$ is nilpotent or $\langle b\rangle$ is nilpotent.

(4) If $\langle a\rangle_l\langle b\rangle_l = 0$, then $\langle a\rangle$ is nilpotent or $\langle b\rangle$ is nilpotent.

Theorem2.14. Let R be a ring and $a, b \in R$ and I is an ideal of R. If R has identity then the following statements are equivalent.

(1) I is a *pgkp*$-$ideal.

(2) If $aRb \subseteq I$, then $\langle a\rangle^k \subseteq I$ or $\langle b\rangle^k \subseteq I$ for some $k \in Z^+$.

(3) If $\langle a\rangle_r \langle b\rangle_r \subseteq I$, then $\langle a\rangle^k \subseteq I$ or $\langle b\rangle^k \subseteq I$ for some $k \in Z^+$.

(4) If $\langle a\rangle_l \langle b\rangle_l \subseteq I$, then $\langle a\rangle^k \subseteq I$ or $\langle b\rangle^k \subseteq I$ for some $k \in Z^+$.

proof. $(1 \Rightarrow 2)$ Let I be a *pgkp*$-$ideal. Since R has identity so we get $\langle a\rangle = RaR$ and $\langle b\rangle = RbR$. As $aRb \subseteq I$, we get $\langle a\rangle R\langle b\rangle = (RaR)R(RbR) \subseteq RaRbR \subseteq RIR \subseteq I$. Now, $\langle a\rangle$ and $R\langle b\rangle$ are ideals of R, so by Theorem 2.8, we have for each $x \in \langle a\rangle$ and $y \in R\langle b\rangle$, there exists $m \in Z^+$ such that $\langle x\rangle^m \subseteq I$ or $\langle y\rangle^m \subseteq I$. As $a \in \langle a\rangle$ and $b = 1.b \in R\langle b\rangle$, so we have $\langle a\rangle^k \subseteq I$ or $\langle b\rangle^k \subseteq I$ for some $k \in Z^+$.

$(2 \Rightarrow 1)$ Suppose that (2) holds and for $a, b \in R$ we have $\langle a\rangle\langle b\rangle \subseteq I$, then we get $aRb \subseteq \langle a\rangle\langle b\rangle \subseteq I$ and so by the given hypothesis there exists $k \in Z^+$ such that $\langle a\rangle^k \subseteq I$ or $\langle b\rangle^k \subseteq I$ and thus I is a *pgkp*$-$ideal.

$(1 \Leftrightarrow 3)$ and $(1 \Leftrightarrow 4)$ can be proved in a similar argument.

By taking $I = 0$ in Theorem 2.14 we give the following corollary.

Corollary2.15. Let R be a ring and $a, b \in R$. If R has identity then the following statements are equivalent.

(1) R is a *pgkp*$-$ring.

(2) If $aRb = 0$, then $\langle a\rangle$ is nilpotent or $\langle b\rangle$ is nilpotent.

(3) If $\langle a\rangle_r\langle b\rangle_r=0$, then $\langle a\rangle$ is nilpotent or $\langle b\rangle$ is nilpotent.

(4) If $\langle a\rangle_l\langle b\rangle_l=0$, then $\langle a\rangle$ is nilpotent or $\langle b\rangle$ is nilpotent.

In the following theorem we prove that in the class of all commutative rings the $pgkp-$rings and $cgkp-$rings are equivalent.

Theorem2.16. If R is a commutative ring then R is a $pgkp-$ring if and only if it is a $cgkp-$ring.

Proof. Suppose that R is a $pgkp-$ring and that $a,b\in R$ such that $ab=0$. Then as R is commutative we get $\langle a\rangle\langle b\rangle=\langle ab\rangle=0$ and since R is a $pgkp-$ring so we get $\langle a\rangle$ is nilpotent or $\langle b\rangle$ is nilpotent. If $\langle a\rangle$ is nilpotent then $\langle a\rangle^n=0$ for some $n\in Z^+$, so $a^n\in\langle a\rangle^n=0$, that means a is nilpotent and if $\langle b\rangle$ is nilpotent by the same argument we get b is nilpotent and thus R is a $cgkp-$ring. Conversely, let R be a $cgkp-$ring and suppose that $c,d\in R$ such that $\langle c\rangle\langle d\rangle=0$. Then as $cd\in\langle c\rangle\langle d\rangle=0$, we have $cd=0$. As R is a $cgkp-$ring we have c is nilpotent or d is nilpotent. If c is nilpotent then $c^k=0$, for some $k\in Z^+$ and then as R is commutative we have $\langle c\rangle^k=\langle c^k\rangle=0$ and so $\langle c\rangle$ is nilpotent and if d is nilpotent by the same argument we get $\langle d\rangle$ is nilpotent. Thus R is a $pgkp-$ring.

Using Theorem 2.4 and Theorem 2.16 we give the following two corollaries.

Corollary2.17. Let R be a ring and I is an ideal of R. If $\frac{R}{I}$ is commutative then I is a $pgkp-$ideal if and only if I is a $cgkp-$ideal.

Proof. We have I is a $pgkp-$ideal if and only if $\frac{R}{I}$ is a $pgkp-$ring if and only if $\frac{R}{I}$ is a $cgkp-$ring if and only if I is a $cgkp-$ideal.

Corollary2.18. Every commutative nil ring is a $pgkp-$ring.

Proof. Since every nil ring is a $cgkp-$ring, the result follows directly.

The converse of Corollary 2.18, is not true in general as we show by the following example.

Let p be a prime number in the ring of integers Z. For $n\in Z^+$, set $\overline{p}=p+\langle p^n\rangle$ in $\frac{Z}{\langle p^n\rangle}$ and $W_n=\langle\overline{p}\rangle$. It is clear that W_n is nilpotent with index n. Let $W=\bigoplus_{n=2}^{\infty}W_n$, then take $A=\bigoplus_{n=1}^{\infty}W_{2n}$ and $B=\bigoplus_{n=1}^{\infty}W_{2n+1}$. Since for $k\neq l$ we have $W_kW_l\subseteq W_k\cap W_l=0$, so we have $AB=0$, where neither A is nilpotent nor B and thus W is not a $gkp-$ring. On the other hand, as W is nil it is a $cgkp-$ring and as it is commutative, by Corollary 2.18, it is a $pgkp-$ring and hence W is a $pgkp-$ring and a $cgkp-$ring which is not a $gkp-$ring.

Theorem2.19. Let R be a ring. If R is a medial $gkp-$ring then it is a $cgkp-$ring.

Proof. Suppose that $x,y\in R$ such that $xy=0$. Since R is medial so we have $xRyR\subseteq xyR=0$ and thus $xRyR=0$ and as R is a $gkp-$ring and xR,yR are right ideals of R, so by Corollary 2.6, we get xR is nilpotent or yR is nilpotent. If xR is nilpotent, then $(xR)^n=0$ for some $n\in Z^+$ and thus $x^{2n}=(x^2)^n=(xx)^n\in(xR)^n=0$ so that $x^{2n}=0$, that means x is nilpotent and if yR is nilpotent then in a similar argument we get y is nilpotent. Hence

R is a $cgkp-$ring.

Theorem2.20. Let R be a ring. If R is a $cgkp-$ring in which all nil ideals are nilpotent then it is a $gkp-$ring.

Proof. Let A and B be two ideals of R with $AB = 0$ and suppose that A is not nilpotent, so by the hypothesis A is not nil which implies that A contains an element a which is not nilpotent. Now, if $b \in B$ is any element, then $ab \in AB = 0$, so that $ab = 0$ and as R is a $cgkp-$ring and a is not nilpotent we must have b is nilpotent. That means every element of B is nilpotent and so B is a nil ideal and hence it is a nilpotent ideal so that R is a $gkp-$ring.

Since a nil left (resp. a nil right) ideal of a left (resp. right) Noetherian ring is nilpotent and also a nil left (resp. a nil right) ideal of a left (resp. right) Artinian ring is nilpotent [1], so by combining these facts with Theorem 2.20 we can give the following corollary.

Corollary2.21. Let R be a ring. If R is a $cgkp-$ring and satisfies any one of the following conditions then it is a $gkp-$ring.

(1) R is a left (resp. a right) Artinian.

(2) R is a left (resp. a right) Noetherian.

(3) R satisfies *acc* on left (resp. right) ideals.

(4) R satisfies *dcc* on left (resp. right) ideals.

Next, we give a condition under which a $pgkp-$ring is a $gkp-$ring.

Proposition2.22. Let R be a ring. If R is a $pgkp-$ring and the sum of any collection of nilpotent principal ideals of R is nilpotent, then R is a $gkp-$ring.

Proof. Suppose that A and B are two ideals of R with $AB = 0$. We have $\sum_{a \in A}\langle a\rangle = A$ and $\sum_{b\in B}\langle b\rangle = B$. If $\langle a\rangle$ is nilpotent for every $a \in A$, then by the hypothesis we have A is nilpotent and the result is obtained and if for some $a \in A$ we have $\langle a\rangle$ is not nilpotent, then for every $b \in B$, we have $\langle a\rangle\langle b\rangle \subseteq AB = 0$ so that $\langle a\rangle\langle b\rangle = 0$ and as R is a $pgkp-$ring and $\langle a\rangle$ is not nilpotent we get $\langle b\rangle$ is nilpotent and as $B = \sum_{b\in B}\langle b\rangle$, by hypothesis we get B is nilpotent. Hence R is a $gkp-$ring.

Corollary2.23. Let R be a ring. If R is a $pgkp-$ring and satisfies any one of the following conditions then it is a $gkp-$ring.

(1) R is a left (resp. a right) Noetherian.

(2) R satisfies *acc* on left (resp. right) ideals.

(3) The sum of any collection of nilpotent ideals of R is nilpotent.

Proof. It is obvious that (1) $\Leftrightarrow$ (2). By [1, page 374], (1) implies (3) and by Proposition 2.22, (3) gives that R is a $gkp-$ring and hence any one of the above conditions gives that R is a $gkp-$ring.

Proposition2.24. An ideal I of a $gkp-$ring R is a $gkp-$ring.

Proof. Let A and B be two ideals of I with $AB = 0$ and let $X = \langle B\rangle_R$ (the ideal of R generated by B). By [4, Proposition 3.1], we get $X^3 \subseteq B$ and thus we have $AX^3 = 0$. Since A is an ideal of I and I is an ideal of R, so we have $AI \subseteq A, ARI \subseteq AI \subseteq A, RAI \subseteq RA$ and $RARI \subseteq RAI \subseteq RA$, thus we get $\langle A\rangle_R IX^3 = (A + AR + RA + RAR)IX^3 = AIX^3 + ARIX^3 + RAIX^3 + RARIX^3 \subseteq AX^3 + AX^3 + RAX^3 + RAX^3 = 0$. Thus $\langle A\rangle_R IX^3 = 0$ and since R is a $gkp-$ring we have $\langle A\rangle_R I$ is nilpotent or X^3 is nilpotent. If $\langle A\rangle_R I$ is nilpotent and as $A \subseteq I$ and $A \subseteq \langle A\rangle_R$, we get $A^2 \subseteq \langle A\rangle_R I$ and hence A is nilpotent and if X^3 is nilpotent and since $B \subseteq X$, we get $B^3 \subseteq X^3$ and thus B^3 is nilpotent, so that B is nilpotent. Hence I is a $gkp-$ring.

It is necessary to mention that the intersection of two $gkp-$ideals of a ring need not be a $gkp-$ideal in general, for example if we take the ideals $\langle 2\rangle$ and $\langle 3\rangle$ in the ring of integers Z, we have $\langle 2\rangle$ and $\langle 3\rangle$ both are $gkp-$ideals of Z, while $\langle 2\rangle \cap \langle 3\rangle = \langle 6\rangle$ is not a $gkp-$ideal of Z.

Theorem2.25. Let R be a ring. If I is an ideal of R and J is a $gkp-$ideal of R then $I \cap J$ is a $gkp-$ideal of I.

Proof. Since J is a $gkp-$ideal of R so by Theorem 2.4, we have $\frac{R}{J}$ is a $gkp-$ring and since $\frac{I+J}{J}$ is an ideal of $\frac{R}{J}$, so by Proposition 2.24, we have $\frac{I+J}{J}$ is a $gkp-$ring and since $\frac{I+J}{J} \cong \frac{I}{I\cap J}$, so $\frac{I}{I\cap J}$ is a $gkp-$ring and again by Theorem 2.4, we have $I \cap J$ is a $gkp-$ideal of I.

In the last two results we give some other properties of $gkp-$rings and $pgkp-$rings.

Proposition2.26. Let R be a $pgkp-$ring. If I is an ideal of R and J is a nonnil right (resp. left) ideal of R such that $I \cap J = 0$, then I is a nil ideal of R.

Proof. We start the proof by considering J as a right ideal and the proof of the case that J is a left ideal can be done similarly. Let $a \in I$ be any element. Then we have $JI \subseteq J \cap I = 0$, that is $JI = 0$. As J is not nil it contains an element say $b \in J$ which is not nilpotent. Now, we have $bRa \subseteq \langle b\rangle_r\langle a\rangle_r \subseteq JI = 0$, so we get $bRa = 0$ and as R is a $pgkp-$ring, by Corollary 2.11, we get $\langle b\rangle$ is nilpotent or $\langle a^2\rangle$ is nilpotent. If $\langle b\rangle$ is nilpotent then we get b is nilpotent which is a contradiction and hence we have $\langle a^2\rangle$ is nilpotent from which we get a is nilpotent. Hence I is a nil ideal of R.

Proposition2.27. Let R be a $gkp-$ring. If I is an ideal of R and J is a nonzero right (resp. left) ideal of R with $I \cap J = 0$, then J is nilpotent or I is nilpotent.

Proof. We have $JI \subset I \cap J = 0$, so $JI = 0$. As R is a $gkp-$ring, so by Corollary 2.6, we have J is nilpotent or I is nilpotent.

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