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Epiform* Modules

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Abstract

Let R be a commutative ring with unity $1 \neq 0$, and let M be a unitary left module over R. In this paper we introduce the notion of epiform* modules. Various properties of this class of modules are given and some relationships between these modules and other related modules are introduced.

Keywords: epiform modules, epiform* modules, coquasi-Dedekind modules, Corational submodules

1 Introduction

Let M be a nonzero R-module, where R is a commutative ring with unity. M is called an epiform module if every proper submodule N of M, $Hom_R(M, \frac{N}{K}) = 0$ for all submodules K of N [1]. Equivalently, M is an epiform module if any homomorphism from M to the factor module of M is an epimorphism [1].

In this paper we introduce a new concept (up to our a knowledgment) namely epiform* module, where an R-module M is called an epiform* if every proper submodule N of M, is either maximal or there exists a proper submodule B of M containing N properly such that $Hom_R(\frac{M}{N}, \frac{B}{X}) = 0$ for all submodules X of M such that $N \leq X \leq B$. In section one of this paper, some basic properties and examples of epiform* modules are given. In section two we investigate some relationships between epiform* modules and other related modules such as epiform, coquasi-Dedekind, copoloyform and copolyform modules (in the sense of [9]). Next in section three, we consider the hereditary property between the ring R and the R-module M

2 Some Basic Properties

We start this section by the following definition:

Definition 2.1. *An R-module M is called an epiform* if every proper submodule N of M, is either maximal or there exists a proper submodule B of M containing N property such that $Hom_R(\frac{M}{N}, \frac{B}{X}) = 0$ for all submodules X of M such that $N \leq X \leq B$.*

Y.Talebi and N.Vanaga in [9] gave the following: For an R-module M and submodules A and B of M such that $A \leq B \leq M$, A is called a corational submodule of B in M, if $Hom_R(\frac{M}{A}, \frac{B}{X}) = 0$ for all submodules X of M such that $A \leq X \leq B$, and it denoted by $A <_{cr} B$ in M.

Remarks and Examples 2.2.

1. *It is clear that every simple module is an epiform* module.*

2. *Semisimple module need not be an epiform* module. For example: Consider the Z-module Z_{10}. Take $N = (\bar{0})$, N is not maximal submodule of Z_{10} and the only proper submodules of Z_{10} containing $< \bar{0} >$ properly are $B_1 =< \bar{0} >$, $B_2 =< \bar{5} >$. But $Hom_R(\frac{Z_{10}}{<\bar{0}>}, \frac{B_1}{<\bar{0}>}) \neq 0$ and $Hom_R(\frac{Z_{10}}{<\bar{0}>}, \frac{B_2}{<\bar{0}>}) \neq 0$.*

3. *The Z-module Z_{p^∞} (for any prime number p) is an epiform*, for if $A < Z_{p^\infty}$, A is not maximal, and for any $B < Z_{p^\infty}$, $A < B$, $Hom_R(\frac{Z_{p^\infty}}{A}, \frac{B}{X}) = 0$ for all submodules X of Z_{p^∞} such that $A \leq X \leq B$, since $Z_{p^\infty} \cong \frac{Z_{p^\infty}}{A}$ and $\frac{B}{X}$ isomorphic to a proper submodules of Z_{p^∞} , and the only homomorphism from Z_{p^∞} in to any proper submodule of Z_{p^∞} is zero.*

4. *For any prime number p, the Z-module Z_{p^3} is not epiform*, since the only nonzero proper submodule of Z_{p^∞} are $< \bar{p} >$, $< \bar{p}^2 >$ and $Hom_R(\frac{Z_{p^3}}{<\bar{0}>}, \frac{<\bar{p}>}{<\bar{0}>}) \neq 0$, $Hom_R(\frac{Z_{p^3}}{<\bar{0}>}, \frac{<\bar{p}^2>}{<\bar{0}>}) \neq 0$.*

Note that by a similar proof of (4), for each prime number p and for each $n \in Z_+$, $n > 1$, the Z-module Z_{p^n} is not epiform*.

Proposition 2.3. *Let M and $\acute{M}$ be two R-modules such that $M \cong \acute{M}$. Then M is an epiform* module if and only if $\acute{M}$ is an epiform* module.*

Proof We have M is an epiform* module, and since $M \cong \acute{M}$, so there exists an isomorphism $f : M \to \acute{M}$. Suppose that $\acute{M}$ is not epiform* module, then $\exists \acute{N} < \acute{M}$ such that $\acute{N}$ is not maximal submodule of M, and for all $\acute{N} < \acute{B} < \acute{M}$,$Hom_R(\frac{\acute{M}}{\acute{N}}, \frac{\acute{B}}{\acute{X}}) \neq 0$ for some submodule $\acute{X}$ of $\acute{M}$ such that $\acute{N} \leq \acute{X} \leq \acute{B}$. That is either $\acute{N}$ is not maximal or $\exists\ g : \frac{\acute{M}}{\acute{N}} \to \frac{\acute{B}}{\acute{X}}$,$g \neq 0$. Since f is an isomorphism, then $\acute{N} = f(f^{-1}(\acute{N}))$, $\acute{B} = f(f^{-1}(\acute{B}))$, $\acute{X} = f(f^{-1}(\acute{X}))$. Put $N = f^{-1}(\acute{N})$, $X = f^{-1}(\acute{X})$, $B = f^{-1}(\acute{B})$. It is clear that $N < M$, $N < B$ and $N \leq X \leq B$. Also $N$is not maximal submodule of M. We claim that$Hom_R(\frac{M}{N}, \frac{B}{X}) \neq 0$. To see that, consider the following diagram:

$$\frac{M}{N} \xrightarrow{h_1} \frac{\acute{M}}{\acute{N}} \xrightarrow{g} \frac{\acute{B}}{\acute{X}} \xrightarrow{h_2} \frac{B}{X}$$

where $h_1(m + N) = f(m) + N$ for each $m + N \in \frac{M}{N}$, $h_2(\acute{b} + \acute{X}) = f^{-1}(\acute{b}) + X$ for each $\acute{b} + \acute{X}$ in$\frac{\acute{B}}{\acute{X}}$ It is easy to check that each of h_1, h_2 is well-defined R-homomorphism. Now $h_2 \circ g \circ h_1 \in Hom_R(\frac{M}{N}, \frac{B}{X})$ since $g \neq 0$, there exists $\acute{m} + \acute{N} \neq 0_{\frac{\acute{M}}{\acute{N}}}$, (so $\acute{m} \notin \acute{N}$ and hence $\acute{m} \neq 0_{\acute{M}}$) such that $g(\acute{m} + \acute{N}) \neq 0_{\frac{\acute{B}}{\acute{X}}}$. But $\acute{m} \in \acute{M}$ and f is an epimorphism,so $\acute{m} = f(m)$ for some $m \in M$. Hence $m \neq 0_M$. Beside that $m \notin N$ because if $m \in N$, then $\acute{m} = f(m) \in f(N) = \acute{N}$ and so $\acute{m} \in \acute{N}$ which is a contradiction. Thus $m + N \neq N = 0_{\frac{M}{N}}$ and $(h_2 \circ g \circ h_1)(m + N) = h_2(g(h_1(m + N)) = h_2(g((f(m) + \acute{N})) = h_2(g(\acute{m} + \acute{N}))$ $= h_2(\acute{b} + \acute{X})$ where $\acute{b} + \acute{X} = g(\acute{m} + \acute{N}) = f^{-1}(\acute{b}) + X$. But $\acute{b} \notin X$and $\acute{b} = f(b)$ for some $b \in B$, so $f(b) \notin f(X)$. Hence $b \notin X$. It follows that: $(h_2 \circ g \circ h_1)(m + N) = b + X \notin X = 0_{\frac{B}{X}}$; that is $h_2 \circ g \circ h_1 \neq 0$ and $Hom_R(\frac{\acute{M}}{\acute{N}}, \frac{\acute{B}}{\acute{X}}) \neq 0$. Thus M is not epiform* module which is a contradiction. Therefore $\acute{M}$ is an epiform* R-module. The proof of the converse is similarly.

Remark 2.4. *A submodule of an epiform* module need not be epiform*, for example: The Z-module Z_{p^∞} is an epiform* module. Let N be a submodule of Z_{p^∞} , where $N =< \frac{1}{p^n} + Z >$ and $n \in Z_+$, $n > 1$. Then $N \cong Z_{p^n}$. But Z_{p^n} is not an epiform* Z-module by Rem. and Ex.(2.2)(4). Hence by Proposition (2.3), N is not an epiform* R-module.*

Proposition 2.5. *Let M be an epiform*R-module and let $N < M$. Then $\frac{\acute{M}}{\acute{N}}$ is an epiform* R-module.*

Proof Let $\frac{A}{N} < \frac{M}{N}$. If $\frac{A}{N}$ is maximal submodule of $\frac{M}{N}$, then nothing to prove. If $\frac{A}{N}$ is not maximal submodule of $\frac{M}{N}$, then A is not maximal submodule of M, so there exists a proper submodule B of M containg A properly such that $A <_{cr} B$ in M. But $N \leq A \leq B$, it follows that $\frac{A}{N} <_{cr} \frac{B}{A}$, because $Hom_R(\frac{\frac{M}{N}}{\frac{A}{N}}, \frac{\frac{B}{A}}{\frac{X}{A}}) \cong Hom_R(\frac{M}{A}, \frac{B}{X})$ for all $A \leq X \leq B$ and $Hom_R(\frac{M}{A}, \frac{B}{X}) = 0$. Thus $\frac{M}{N}$ is an epiform*.

Corollary 2.6. *The epimorphic image of epiform* module is epiform*.*

Proof Let $f : M \to \acute{M}$ be an R-epimorphism and let M be an epiform* R-module.By 1^{st} Fundamental theorem,$\frac{M}{kerf} \cong \acute{M}$. But $\frac{M}{kerf}$ is an epiform* R-module by Proposition (2.5) . Hence $\acute{M}$ is an epiform* module by Proposition (2.3).

Corollary 2.7. *A direct summand of an epiform* module is an epiform* submodule.*

Proof Let M be an epiform* module and N be a direct summand of M. Hence $M = N \oplus K$ for some $K < M$. Then $\frac{M}{K} \cong N$. And by Proposition (2.5), $\frac{M}{K}$ is an epiform* module. Thus N is an epiform* module by Proposition (2.3).

Next we have the following remark.

Remark 2.8. *A direct sum of epiform* R-modules need not be epiform* module, as the following example shows: Each of the Z-module Z_2, Z_5 is an epiform* module but $Z_2 \oplus Z_5$ is not epiform* Z-module, since by Rem. and Ex. (2.2)(2), $Z_2 \oplus Z_5 \cong Z_{10}$ and Z_{10} is not an epiform* Z-module*

3 Epiform* Modules and Related Modules

In this section we give some relationships between epiform* module and some other related modules such as epiform, coquasi-Dedekind, copolyform, copolyform (in the sense of [9]) and antihopfian modules. Firstly, we need to give the following.

Lemma 3.1. *Let M be an R-module. If $\frac{M}{N}$ is an epiform R-module for all proper submodules N of M, then M is an epiform* R-module.*

Proof Let N be a proper submodule of M. Suppose that N is not maximal submodule of M, so there exists a proper submodule B of M such that $N < B$. Hence $\frac{B}{N} < \frac{M}{N}$. Since $\frac{B}{N}$ is an epiform R-module, then $Hom_R(\frac{M}{N}, \frac{\frac{B}{N}}{\frac{X}{N}}) = 0$ for all submodules $\frac{X}{N}$ such that $\frac{X}{N} \leq \frac{B}{N}$. It follows that $Hom_R(\frac{M}{N}, \frac{B}{X}) = 0$ for all submodules X of M, such that $N \leq X \leq B$. Thus M is an epiform* R-module.

Note that the Z-module Z_4 is not epiform. However $\frac{Z_4}{<2>} \cong Z_2$ is an epiform* module.

The following proposition related epiform* with epiform modules.

Proposition 3.2. *Let M be an epiform R-module. Then M is an epiform* R-module.*

Proof Since M is epiform, then by [1], $\frac{M}{N}$ is an epiform for all $N < M$. Hence by Lemma (3.1), M is an epiform* module.

Notes 3.3.

1. *It is clear that an R-module M is an epiform if and only if $(0) <_{cr} B$ in M, for all $B < M$. However M is an epiform* R-module implies that $(0) <_{cr} B$ in M, for some $B < M$ or (0) is a maximal submodule of M. Hence we claim that epiform* module is not necessarily epiform module, but we have no example to ensure this.*

2. *Every nonzero Artinain module M has an epiform submodule by [1], hence M has an epiform* submodule by Proposition (3.2).*

Recall that an R-module M is called a coquasi-Dedekind if $Hom_R(M, N) = 0$ for each proper submodule N of M. Equivalently any nonzero $f \in End(M)$ is an epimorphism [11]. It is clear that every epiform module is coquasi-Dedekind module.

We can give the following example.

Example 3.4. *Consider the Z-module Q. By ([11], Ex.2.1.8), Q is a coquasi-Dededkind Z-module and $\frac{Q}{Z}$ is not coquasi-Dedekind Z-module. Hence $\frac{Q}{Z}$ is not an epiform Z-module. Also Q is not an epiform Z-module because if it is epiform, then by [1], $\frac{Q}{Z}$ is an epiform Z-module which is a contradiction.*

Recall that an R-module M is called multiplication if for each $N \leq M$, there exists an ideal I of M such that $N = IM$. Equivalently M is a multiplication R-module if for each $N \leq M$, $N = (N :_R M)M$ [2].

Proposition 3.5. *Let M be a multiplication (or projective or finitely generated) R-module. Consider the following statements*

1. *M is an epiform module.*

2. *M is a coquasi-Dedekind module.*

3. *M is an epiform* module.*

Then (1)⇔(2) ⇒ (3).

proof (1) ⇒ (2): It is clear
(2) ⇒ (1): By ([11], Cor. 2.3, Prop. 2.2.18 and Cor. 2.3.7), M is simple module. Hence M is an epiform module.
(1) ⇒(3) it follows from Proposition (3.2).

Recall that an R-module is called selfgenerator if for every submodule N of M, $N = \sum_f Imf$, where $f \in Hom_R(M, N)$ ([7],p.241)

Proposition 3.6. *Let M be a selfgenerator R-module. Consider the following statements*

1. *M is an epiform module.*

2. *M is a coquasi-Dedekind module.*

3. *M is an epiform* module.*

Then (1)⇔(2) ⇒ (3)

proof (1) ⇒ (2): it is clear
(2) ⇒ (1), (2)⇒ (3) By ([11], Prop. 2.3.5), every selfgenerator coquasi-Dedekind module is simple, so M is an epiform and epiform* module.

Recall that an R-module M is called antihopfian if $\frac{M}{N} \cong M$ for all submodules N of M such that $N < M$ [6]. We have the following:

Proposition 3.7. *Let M be an antihopfian R-module. Then M is an epiform module and hence it is an epiform* module.*

Proof Let $N < M$. Since M is antihopfian, then N is antihopfian module [6]. Hence $\frac{N}{K} \cong N$ for all submodules K such that $K < N$. It follows that $Hom_R(M, \frac{N}{K}) = Hom_R(M, N)$ for all $K < N$. But by ([11], Prop. 2.3.3), every antihopfian module is coquasi-Dedekind. Thus $Hom_R(M, N) = 0$ and so $Hom_R(M, \frac{N}{K}) = 0$ for all $K < N$; that is M is an epiform module and hence epiform* module.

Now we have the following

Proposition 3.8. *Let M be a coquasi-Dedekind module such that for each $N < M$, N is antihopfian. Then M is an epiform module and hence it is epiform* module.*

Proof Let $N < M$. Suppose N is not maximal submodule of M. Since N is antihopfian, then $\frac{N}{K} \cong N$ for all submodules $K < N$ and since M is coquasi-Dedekind, so $Hom_R(M, N) = 0$. It follows that $Hom_R(M, \frac{N}{K}) = 0$ $\forall$ $K < N$. Thus M is an epiform module and hence M is an epiform* module.

Recall that if $A \leq B \leq M$, A is called coessential submodule of B in M (denoted by $A <_{ce} B$ in M) if B/A is small submodule of M/A (i.e. $B/A \ll M/A$) [10].

Proposition 3.9. *Let M be an epiform* module. Then every proper submodule A of M, is either maximal submodule of M or $A <_{ce} B$ in M for some proper submodule B of M containing A properly.*

Proof Let $A < M$. Since M is an epiform*, then either A is maximal submodule of Mor there exists a proper submodule B of M containing A properly such that $A <_{cr} B$ in M. And by [9], $A <_{ce} B$ in M.

The converse of proposition (3.9) is not true in general, for example the Z-module Z_4 is not an epiform*, but Z_4 has only two proper submodules $< \bar{0} >, < \bar{2} >$, , where $< \bar{0} > <_{ce} < \bar{2} >$ in Z_4, and $< \bar{2} >$ is maximal submodule of Z_4. However, the converse is true if a module M is noncosingular, where an R-module M is called noncosingular, if for any nonzero R-module N and for every nonzero R-homomorphism $f : M \to N$, Imf is not small submodule of$N$ [4].

Proposition 3.10. *Let M be a noncosingular module. Then M is an epiform* module if and only if for every proper submodule N of M, N is either maximal or $N <_{ce} B$ in M for some proper submodule B of M.*

Proof $\Rightarrow$It follows from Proposition(3.9)

$\Leftarrow$ It follows directly from [9].

Recall that an R-module M is called copolyform if $Hom_R(M, \frac{X}{A}) = 0$ for all submodules A and X such that $A \leq X \leq M$, [8]

Y. Talebi and N.Vanaja in [9] defined copolyform module in a different way as follows: A module M is called copolyform if whenever $B/A \ll M/A$, then $Hom_R(\frac{M}{A}, \frac{B}{X}) = 0$ for all submodules A and X of M such that $A \leq X \leq B$. She proved that M is a copolyform module (in the sense of [9]), if and only if $\frac{M}{N}$ is copolyform module for every submodule N of M. Thus for an R-module M, it can be easily show the following statements are equivalent:

1. M is an epiform module.

2. M is Hollow and copolyform module.

3. M is Hollow and copolyform module(in the sense of [9])

where M is called Hollow module if $N \ll M$, for all $N < M$ [5]. Hence every Hollow copolyform module(or copolyform in the sense of [9]) is epiform*, but copolyform module (or copolyform in the sense of [9]) need not be epiform*. For example, the Z-module Q is copolyform module, but it is not an epiform* Z-module. Also Z_6 as Z-module is copolyform (in the sense of [9]), hence copolyform, but it is not an epiform* module.

Next we have the following.

Proposition 3.11. *Let M be a copolyform module (in the sense of [9]) and for each $N < M$, either N is maximal submodule of M or there exists a proper submodule B of M containing N properly such that $N <_{ce} B$ in M. Then M is an epiform* module.*

Proof Let $N < M$. If N is not maximal submodule of M, so there exists a proper submodule B of M such that $N < B$ and $N <_{ce} B$ in M; that is $\frac{B}{N} \ll \frac{M}{N}$. But M is a copyform module (in the sense of [9]), so $Hom_R(\frac{M}{N}, \frac{\frac{B}{N}}{\frac{X}{N}}) = 0$ for all submodules $\frac{X}{N}$, $\frac{B}{N}$ such that $\frac{X}{N} \leq \frac{B}{N}$. It follows that $Hom_R(\frac{M}{N}, \frac{B}{X}) = 0$ for all submodules N, such that $N \leq X \leq B$.Thus M is an epiform* module.

4 Hereditary of Epiform* Modules

In this section we study the hereditary property of the class of epiform* modules, that is the transitivity of epiform* property between an R-module M and the ring R which defined on it. Firstly we start by the following theorem.

Theorem 4.1. *Let M be a finitely generated faithful multiplication R-module. If M is an epiform* R-module then R is an epiform* R-module*

Proof Let I be a proper ideal of R such that I is not maximal. Suppose that I is not corational ideal of any proper ideal J of R containing I properly; that is $(\frac{R}{I}, \frac{J}{K}) \neq 0$ for some ideal $I \leq K \leq J$. Let$f \in Hom_R(\frac{R}{I}, \frac{J}{K})$, $f \neq 0$. Hence $f(1 + I) \neq K = 0_{J/K}$. Put $f(1 + I) = j + K$ for some $j \in J$ and $j \neq K$. Since M is a finitely generated faithful multiplication module, so $IM < JM < M$. Define $g : \frac{M}{IM} \to \frac{JM}{KM}$ by $g(m + IM) = jm + KM$ for all $m \in M$. It is clear that g is well-defined R- homomorphism. But M is an epiform* R-module, so g must be zero mapping. Hence $jm+KM = KM$ for all $m \in M$, and this implies that $jm \in KM$ for all $m \in M$. Thus $(j)M \leq KM$. But M is a finitely generated faithful multiplication R-module, thus $(j) \leq K$; that is $j \in K$ which is a contradiction. Therefore R must be an epiform* R-module.

Proposition 4.2. *Let M be a finitely generated faithful multiplication over a principle ideal ring R. If R is an epiform* R-module, then M is an epiform* R-module.*

Proof Let N be a proper submodule of M. Then $N = IM$ for some proper ideal I of R. Assume that N is not maximal submodule of M and for all submodule W contains N properly, there exists a nonzero homomorphism $f : \frac{M}{N} \to \frac{W}{X}$, for some submodule X, such that $N \leq X \leq W$. But $W = JM$, $X = LM$ for some ideals J and L of R. Since $N < W$, then $I < J < R$ by ([3], Th.3.1). Since $f \neq 0$, so there exists $m \in M$, $m \notin N = IM$ such that $f(m + N) \neq 0_{W/X}$. Now assume that $f(m + N) = \acute{m} + X$for some $\acute{m} \in W = JM$. But J is a principal ideal then $\exists\ a \in J$, $J =< a >$ and hence $\acute{m} = am_1$ for some $m_1 \in M$ and $\acute{m} = am_1 \notin X = LM$. Now define $f : \frac{R}{I} \to \frac{J}{L}$ by $f(r + I) = ra + L$ for each $r \in R$. It is clear that f is well-defined R-homomorphism, but R is an epiform*, thus $f = 0$. It follows that $ra \in L$ for all $r \in R$ and hence $ram_1 \in LM$, and hence $am_1 \in LM$ which is a contradiction. Thus M is an epiform* module.

Corollary 4.3. *Let M be a faithful finitely generated multiplication R-module. Consider the following statements*

1. *M is an epiform* R-module.*
2. *M is a coquasi-Dedekind R-module.*
3. *M is an epiform* R-module.*
4. *R is an epiform* R-module.*
5. *R is a field.*
6. *R is a coquasi-Dedekind R-module.*
7. *R is an epiform R-module.*

Then (1)$\Leftrightarrow$(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Leftarrow$ (5) $\Leftrightarrow$ (6)$\Leftrightarrow$(7), and if R is a principal ideal ring, then by proposition (4.2) we get (4)$\Rightarrow$ (3).

Proof (1)$\Leftrightarrow$ (2) $\Rightarrow$ (3) It follows from Proposition (3.5).
(3) $\Rightarrow$ (4) It follows from Theorem (4.1).
(5) $\Rightarrow$ (4) R is a field, so it is simple R-module. Hence R is an epiform* R-module.
(5)$\Leftrightarrow$(6) It is clear.
(6) $\Rightarrow$ (7) R is a coquasi-Dedekind R-module and R is a multiplication R-module. So R is simple module. Thus R is an epiform R-module.
(7) $\Rightarrow$ (6) It is clear.

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