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# Lindstedt Perturbation Analysis of a Time Delay Economic Model Near a Hopf Bifurcation

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**Abstract**

In this paper, the Hopf bifurcation of a delayed Solow-Swan type model is studied by using Lindstedt's perturbation method. The limit cycle is showed to be stable and the Hopf bifurcation to be supercritical.

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**Keywords:** Time delay; Hopf bifurcation; Lindstedt method

## 1 Introduction

Guerrini [2] had studied the effect of the time delay on the stability of a standard neoclassical one-sector growth model [3-4], where the two types of agents, workers and shareholders, have different but constant saving rates. New capital is not to produce instantaneously but it is produced and installed after $T$ periods. The resulting capital accumulation law in Guerrini [2] is therefore given by

$$\dot{k}(t) = [s_w + \alpha(s_c - s_w)]k(t-T)^\alpha - \delta k(t-T), \tag{1}$$

for some initial function $k(t) = \phi(t)$, $t \in [-T, 0]$. Here, $k(t)$ denotes physical capital, $\delta$ is the depreciation rate of capital, $\alpha \in (0,1)$ represents capital's share, $s_w$ and $s_c$ are the constant saving rates for workers and shareholders respectively, $s_w < s_c$. The constant labor force growth rate $n$ is assumed to be normalized to unity ($n = 0$, $L(t) = 1$). It was showed that the system loses stability and a Hopf bifurcation occurs when time delay passes through the critical value $T_0 = \pi/(2\omega_0)$, with $\omega_0 = (1-\alpha)\delta$. But neither the direction and stability of the local Hopf bifurcation were considered in his paper. In this paper, to gain insight of the bifurcated periodic solution, the Lindstedt method

[1] is used to investigate the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions. It is found that the Hopf bifurcation is supercritical and the bifurcating solutions are stable.

## 2 Direction and stability of Hopf bifurcation

Let $k_*$ be the unique positive equilibrium point of (1). Then $k_*$ satisfies $[s_w + \alpha(s_c - s_w)]k_*^{\alpha-1} = \delta$. Define $x_t = k_t - k_*$. Taking a Taylor expansion of Eq. (1), including the linear, quadratic and cubic terms, yields

$$\dot{x}(t) = -(1-\alpha)\delta x\left(t-T\right) - \frac{(1-\alpha)\alpha\delta k_*^{-1}}{2} x\left(t-T\right)^2 + \frac{(1-\alpha)(2-\alpha)\alpha\delta k_*^{-2}}{6} x\left(t-T\right)^3 + \cdots . \quad (2)$$

We introduce a small parameter via the scaling $s = \omega\left(\varepsilon\right)t$, where $\omega$ is a parameter close to $\omega_0$ and $\varepsilon$ is a small positive number. In this way, the solutions which are $2\pi/\omega$ periodic in $t$ become periodic with period $2\pi$. Next, we rewrite Eq. (2) as

$$\omega\frac{dx(s)}{ds} = -(1-\alpha)\delta x\left(s-\omega T\right) - \frac{(1-\alpha)\alpha\delta k_*^{-1}}{2} x\left(s-\omega T\right)^2 + \frac{(1-\alpha)(2-\alpha)\alpha\delta k_*^{-2}}{6} x\left(s-\omega T\right)^3 + \cdots . \quad (3)$$

We expand the periodic solution of Eq. (3) in a power series of $\varepsilon$:

$$x(s,\varepsilon) = \varepsilon x_0(s) + \varepsilon^2 x_1(s) + \varepsilon^3 x_2(s) + \cdots , \quad (4)$$

with the obvious definition of $x_0, x_1, ...$ Similarly, the coefficients $\omega$ and $T$ are expanded as

$$\omega = \omega\left(\varepsilon\right) = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots , \quad T = T\left(\varepsilon\right) = T_0 + \varepsilon T_1 + \varepsilon_2^2 T_2 + \cdots . \quad (5)$$

Next, using (4) and (5), we expand the delay term $x\left(s-\omega\tau\right)$ :

$$x\left(s-\omega T\right) = \varepsilon x_0(s-\omega T) + \varepsilon^2 x_1(s-\omega T) + \varepsilon^3 x_2(s-\omega T) + \cdots ,$$

where

$$x_j(s-\omega T) = x_j(s-\omega_0 T_0) - x_j'(s-\omega_0 T_0)\left[\varepsilon(\omega_1 T_0 + \omega_0 T_1) + \varepsilon^2(\omega_2 T_0 + \omega_1 T_1 + \omega_0 T_2) + \cdots\right] + \frac{1}{2} x_j''(s-\omega_0 T_0)\left[\varepsilon(\omega_1 T_0 + \omega_0 T_1) + \cdots\right]^2 - \cdots .$$

Substituting and collecting terms, we obtain the equations governing the terms $x_0(s), x_1(s)$ and $x_2(s)$ :

$$\frac{dx_0(s)}{ds} = -x_0\left(s - \frac{\pi}{2}\right). \tag{6}$$

$$\begin{aligned}(1-\alpha)\delta\,\frac{dx_1(s)}{ds} &+ (1-\alpha)\delta x_1\left(s-\frac{\pi}{2}\right)\\ &= -\omega_1\frac{dx_0(s)}{ds} + (1-\alpha)\delta x_0'\left(s-\frac{\pi}{2}\right)\left[\omega_1 T_0 + (1-\alpha)\delta T_1\right]\\ &\quad - \frac{(1-\alpha)\alpha\delta k_*^{-1}}{2}\,x_0\left(s-\frac{\pi}{2}\right)^2.\end{aligned} \tag{7}$$

$$\begin{aligned}(1-\alpha)\delta\frac{dx_2(s)}{ds} &+ (1-\alpha)\delta x_2\left(s-\frac{\pi}{2}\right)\\ &= -\omega_2\frac{dx_0(s)}{ds} + (1-\alpha)\delta x_0'\left(s-\frac{\pi}{2}\right)\left[\omega_2 T_0 + (1-\alpha)\delta T_2\right]\\ &\quad - (1-\alpha)\alpha\delta k_*^{-1}x_0\left(s-\frac{\pi}{2}\right)x_1\left(s-\frac{\pi}{2}\right)\\ &\quad + \frac{(1-\alpha)(2-\alpha)\alpha\delta k_*^{-2}}{6}\,x_0\left(s-\frac{\pi}{2}\right)^3.\end{aligned} \tag{8}$$

We take the solution of Eq. (6) as

$$x_0(s) = A\sin s. \tag{9}$$

Next, we seek the general solution of (7) in the form $x_1(s) = A_1\sin s + B_1\cos s + C_1\sin(2s) + D_1\cos(2s) + E_1$. Substituting (9) into Eq. (7) and equating the coefficients of $\sin s, \cos s, \sin(2s), \cos(2s)$, we obtain

$$\omega_1 = T_1 = 0, \qquad C_1 = 2D_1, \qquad D_1 = -\frac{\alpha k_*^{-1}A^2}{20}, \qquad E_1 = 5D_1.$$

For simplicity, we impose $A_1 = B_1 = 0$. Next, we substitute $x_0(s)$ and $x_1(s)$ into Eq. (8), and, after trigonometric simplifications have been performed, we equate to zero the coefficients of the resonant terms *sin* and *cos*. This yields the amplitude, $\varepsilon A$, of the limit cycle that was born in the Hopf bifurcation:

$$(\varepsilon A)^2 = \frac{P}{Q}T_2, \tag{10}$$

where

$$P = 20(1-\alpha)^7\delta^7 > 0, \qquad Q = \frac{(1-\alpha)^6\alpha\delta^6 k_*^{-2}\left[(3\alpha+5)\pi - 2\alpha\right]}{2} > 0.$$

We see from (10) that $T_2$ must have the same sign as $P/Q$. From the previous discussion, we may immediately obtain the following result.

**Theorem 2.1.** *The limit cycle is stable and we have a supercritical Hopf bifurcation.*

# References

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