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Congruences in Hypersemilattices

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Abstract

In this paper we study congruence relation of hypersemilattices and Homomorphism and isomorphism of hypersemilattices using congruence relation and we prove that hyper meet of two congruence relations is a Congruence relation and is a fixed element of hypersemilattices. Also we prove embedding theorem for hypersemilattices using congruence relation. Finally we prove theorem on family of direct product of hypersemilattices

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Introduction

The Theory of hyperstructures was introduced in1934 by Marty [1] at the 8th congress of Scandivinavian Mathematicians. This theory has been subsequently developed by the various authors. Some basic definitions and propositions about the hyperstructures are found in[3].Throughout this paper we are using definitions of hypersemilattice as discussed in [4].In this paper the concepts of congruence relation are discussed, Also we relate this to isomorphism and homomorphism, that is we prove some results on hypersemilattices using congruence relation.

1. Preliminaries

Definition 1.1 [4]: Let L be a non-empty set and let P(L) denote the Power set of L,P*(L)= P(L)-{ φ}.A binary operation hyperoperation "o" on L is a function from L x L to P*(L) and satisfies the following conditions.
For all a, b, c Є L and all A,B,C Є P*(L) we have that a o b Є P*(L) ,C o A = $\underset{a\epsilon A}{U}$(c o a) Є P*(L), A o C = $\underset{a\epsilon A}{U}$ (a o c) Є P*(L) , A o B = $\underset{a\epsilon A\,,\,b\epsilon B}{U}$(a o b) Є P*(L) .

Definition 1.2 [3]: Let L be a non-empty set and ⊕ : L x L→ P(L) be a hyperoperation ,where P(L) is a power set of L and P*(L) =P(L) -{φ} and ⊗ : L x L → L be an operation. Then (L, ⊗ ⊕) is a hyperlattice if for all a, b ,c Є L.

1. a Є a ⊕ a, a ⊗ a=a
2. a ⊕b=b ⊕ a, a ⊗ b=b ⊗ a
3. (a ⊕b) ⊕c=a ⊕(b ⊕c) ,(a⊗ b) ⊗ c=a ⊗(b⊗ c).
4. a Є [a⊗(a ⊕ b)] ∩[a ⊕ a ⊗ b]
5. a Є a ⊕ b a ⊗b=b.

Definition 1.3[4] : Hypersemilattices: Let L be a non-empty set with a hyper operation ⊗ On L satisfying the following conditions, for all a, b, c Є L

1. a Є a ⊗ a (Idempotent)
2. a ⊗ b = b ⊗ a (Commutative)
3. (a ⊗ b) ⊗ c = a⊗ (b ⊗ c). (Assosiative)

Then (L, ⊗) is called a hypersemilattice.

Definition 1.4 [4]: Let (L, ⊗) be a hypersemilattice. An element a Є L is called absorbent element of L if it satisfies c Є a ⊗ c for all c Є L. An element b Є L is called element of L if it satisfies b ⊗ c = {b} for all c Є L.

Proposition 1.5 [4]: Let (L, ⊗) be a hypersemilattice,then a ⊗ c ⊆ a ⊗ (a ⊗ c) for all a, c Є L.

Definition 1.5 [4]: Let (L, ⊗) and (S, ⊗) be a hypersemilattices. A function f: L → S is called a homomorphism provided f(a ⊗ b) = f(a) ⊗ f(b) for all a, b Є L. If f is injective as a map of sets, f is said to be a Monomorphism. If f is surjective, f is called Epimorphism. If f is bijective, f is called an isomorphism.

2. Homomorphism and congruence of Hypersemilattices

Theorem 2.1: The homomorphic Image of a hypersemilattice is also a hypersemilattice.

Proof: Let f: L → S be a homomorphism on L
Let $a_1, b_1, c_1 \in f(L)$,Then ∋ a, b, c ∈ L such that $f(a) = a_1$, $f(b) = b_1$, $f(c) = c_1$. Since L is a hypersemilattice we have ⇒ $a \in a \otimes a \Rightarrow f(a) \in f(a \otimes a)$ $f(a) \in f(a) \otimes f(a)$ $a_1 \otimes b_1 = f(a) \otimes f(b) = f(a \otimes b) = f(b \otimes a) = f(b) \otimes f(a) = b_1 \otimes a_1$, $(a_1 \otimes b_1) \otimes c_1 = [f(a) \otimes (b)] \otimes f(c) = f[(a \otimes b) \otimes c] = f[a \otimes (b \otimes c)] = f(a) \otimes [f(b) \otimes f(c)] = a_1 \otimes (b_1 \otimes c_1)$,
Hence f(L) is a hypersemi lattice. □

Definition 2.2 : Let <L, ⊗ > be a hypersemilattice and θ be an equivalence relation on L. We say that, A θ B if and only if for all a ∈ A ∋ b ∈ B such that a θ b for any A,B ⊆ L .

Definition 2.3: Let L be hypersemilattice, θ is said to be congruence relation if for any a, b, c, d ∈ L, a θ b and c θ d imply $(a \otimes c)\ \theta\ (b \otimes d)$. We denote the equivalence class $\{y \in L / x\ \theta\ y\}$ by C_x for x ∈ L.

Theorem 2.4: If θ is a congruence relation on L then, <L/θ,⊗> is a hypersemilattice.

Proof: Define the hyperoperation ⊗ on L/ θ as $C_x \otimes C_y = \{c_t / t \in x \otimes y\}$. Since L is a hypersemilattice, $x \in x \otimes x$ and hence $C_x \in C_x \otimes C_x$, Since $x \otimes y = y \otimes x$, we have $C_x \otimes C_y = C_y \otimes C_x$, Note that $(C_x \otimes C_y) \otimes C_z = \bigcup_{t \in x \otimes y} \{ C_p / p \in t \otimes z \}$ and $C_x \otimes (C_y \otimes C_z) = \bigcup_{r \in y \otimes z} \{ C_q / q \in x \otimes r\}$. Let $C_s \in (C_x \otimes C_y) \otimes C_z)$ then $C_s \in \{ C_p / p \in t \otimes z \}$ for some $t \in x \otimes y \Rightarrow s \in t \otimes z$ for some $t \in x \otimes y \Rightarrow s \in (x \otimes y) \otimes z$ $\Rightarrow C_s \in \{ C_q / q \in x \otimes r \}$ for some $r \in y \otimes z \Rightarrow C_s \in \bigcup_{r \in y \otimes z} \{ C_q / q \in x \otimes r \} = C_x \otimes (C_y \otimes C_z)$ and hence $(C_x \otimes C_y) \otimes C_z) \subseteq C_x \otimes (C_y \otimes C_z)$. Similarly we can prove $C_x \otimes (C_y \otimes C_z) \subseteq (C_x \otimes C_y) \otimes C_z)$ .Therefore $(C_x \otimes C_y) \otimes C_z) = C_x \otimes (C_y \otimes C_z)$. L/ θ is a hypersemilattice.

□

Proposition 2.5: $[\, C_x \otimes C_y]_{\Phi/\theta} = (C_x)_{\Phi/\theta} \otimes (C_y)_{\Phi/\theta}$.

Proof: Let $C_t \in \{ [\, C_x \otimes C_y]_{\Phi/\theta} \,/\, t \in (x \otimes y)_\Phi \} \Leftrightarrow t \in (x \otimes y)_\Phi \Leftrightarrow t \in (x)_\Phi \otimes (y)_\Phi \Leftrightarrow C_t \in \{ (C_x)_{\Phi/\theta} \otimes (C_y)_{\Phi/\theta} / t \in (x)_\Phi \otimes (y)_\Phi \} \Leftrightarrow [\, C_x \otimes C_y]_{\Phi/\theta} = (C_x)_{\Phi/\theta} \otimes (C_y)_{\Phi/\theta}$. □

Theorem 2.6: Let θ be a congruence relation defined on L then L/θ is a homomorphic image of the hypersemilattice L.

Proof: Let $f : L \to L/\theta$ be defined by $f(x_\Phi) = [C_x]_{\Phi/\theta}$ .Obviously f is well-defined and to show f is homomorphism, consider $f(x_\Phi \otimes y_\Phi) = [C_x \otimes C_y]_{\Phi/\theta} = (C_x)_{\Phi/\theta} \otimes (C_y)_{\Phi/\theta} = f(x_\Phi) \otimes f(y_\Phi)$. Therefore f is homomorphism. Select any $[C_x] \in \Phi/\theta$, then $x \in \Phi$. For this $x \in \Phi$, $f(x_\Phi) = [C_x]_{\Phi/\theta}$ .f is surjective. Thus there exist an onto homomorphism from L to L/θ. Hence L/θ is homomorphic image of L. □

Definition 2.7: Let f be a homomorphism from a hypersemilattice L_1 to a hypersemilattice L_2.Then Ker f = $\{x, y \in L_1 / f(x) = f(y)\}$.

Theorem 2.8: Kernel f is a congruence relation on hypersemilattice L.

Proof: First we have to prove kerf is an equivalence relation. As $x \,\theta\, y \Rightarrow f(x) = f(y)$. Obviously kerf is an equivalence relation. We prove Kerf is a congruence relation .Let $x \,\theta\, y \Rightarrow f(x)=f(y)$ and $a \,\theta\, b \Rightarrow f(a)=f(b)$. Let us consider $f(x \otimes a) = f(x) \otimes f(a) = f(y) \otimes f(b) = f(y \otimes b)$. Therefore Ker f is a congruence relation. □

Theorem 2.9: If $h: L_1 \to L_2$ is a surjective homomorphism then there exist an isomorphism $f: L_1/\ker h \to L_2$.

Proof: Define $f: L_1/\ker h \to L_2$ by $f(C_x) = h(x)$,where C_x is an equivalence class of x under ker h. Clearly f is well-defined. Let $C_x, C_y \in L_1/\ker h$ such that $f(C_x)=h(C_y)$.Then $h(x)= h(y) \Rightarrow (x, y) \in$ Ker f $\Rightarrow C_x = C_y$ and hence $h(x)=h(y)$ and hence f is one-one. To prove homomorphism, Let $f(C_x \otimes C_y) = f\{C_t / t \in x \otimes y\} = \{h(t) / t \in x \otimes y\} = h(x \otimes y) = h(x) \otimes h(y) = f(C_x) \otimes f(C_y)$.Hence f is homomorphism. Since h is onto and for any $y \in L_2$ there exist $C_x \in L_1/\ker h$ such that $f(C_x) = h(x) = y$. Hence f is onto. Therefore, $L_1/\ker h \cong L_2$. □

Theorem 2.10: product of two congruence relations is a congruence relation.

Proof: Let θ and Φ be any congruence relations defined on Hypersemilattices L and K respectively. Define the relation Ψ on L x K by $(C_a \otimes C_b)\ \Psi\ (C_c \otimes C_d) \Rightarrow C_a\ \theta\ C_c$, $C_b\ \Phi\ C_d$. Then we prove that Ψ is a congruence relation defined on L x K . To prove that $(C_a \otimes C_b)\ \Psi(C_a \otimes C_b)$, as a θ a and $C_b\ \Phi\ C_b$. Therefore $(C_a \otimes C_b)\ \Psi(C_a \otimes C_b)$. Ψ is reflexive. Let $(C_a \otimes C_b)\ \Psi(Cc \otimes Cd)$, then $C_a\ \theta\ C_c$ and $C_b\ \Phi\ C_d$ but θ and Φ and are equivalence relations $C_c\ \theta\ C_a$ and $C_d\ \Phi\ C_b$. therefore $(C_c \otimes C_d)\ \Psi\ (C_a \otimes C_b)$. Ψ is symmetric, Let $(C_a \otimes C_b)\ \Psi\ (C_c \otimes C_d)$ and $(C_c \otimes C_d)$ and $(C_x \otimes C_y)$then $C_a\ \theta\ C_c$ and $C_c\ \theta\ C_x$, $C_b\ \Phi\ C_d$ and $C_d\ \Phi\ C_y$ by transitivity of θ and Φ , $C_a\ \Phi\ C_x$ and $C_b\ \Phi\ C_y$. $(C_a \otimes C_b)\ \Psi\ (C_x \otimes C_y)$.Therefore is transitive. Let $(C_{a1} \otimes C_{b1})\ \Psi\ (C_{c1} \otimes C_{d1})$ and $(C_{a2} \otimes C_{b2})\ \Psi\ (C_{c2} \otimes C_{d2})$ This implies $C_{a1}\ \theta\ C_{c1}$ and $C_{b1}\ \Phi\ C_{d1}$ and $C_{a2}\ \theta\ C_{c2}$ and $C_{b2}\ \Phi\ C_{d2}$.Hence $(C_{a1} \otimes C_{a2})\ \theta\ (C_{c1} \otimes C_{c2})$ and $(C_{b1} \otimes C_{b2})\ \Phi\ (C_{d1} \otimes C_{d2})$.Therefore $(C_{a1} \otimes C_{a2}) \otimes (C_{b1} \otimes C_{b2}) = (C_{c1} \otimes C_{d1}) \otimes (Cc_2 \otimes Cd_2)$ .Therefore Ψ is a congruence relation de fined on L x K. □

Definition 2.11 : If θ and Φ are congruence relations on L with $\theta \subseteq \Phi$ then define a relation Φ/θ on L/θ by $(C_x , C_y) \in \Phi/\theta$ if and only if $(x , y) \in \Phi$.

Theorem 2.12: Every congruence relation is the kernel of some homomorphism.

Proof: Let Φ is a congruence relation on L. Clearly Φ is a Equivalence relation on L/θ, then to prove Φ/θ is a congruence relation .Let (C_x , C_y) and $(C_z , C_w) \in \Phi/\theta \Rightarrow$ (x, y) ,$(z, w) \in \Phi$, This implies $(x \otimes z)$,$(y \otimes w) \in \Phi/\theta$ .That is $(C_{x \otimes z} , C_{y \otimes w})$ Hence Φ/θ is congruence relation on L/θ. Now let Φ/θ is a congruence relation on L/θ. Similarly we can prove Φ is a congruence relation on L.As Φ is a congruence relation on L this implies $(x ,y) \in \Phi \Leftrightarrow (C_x , C_y) \in \Phi/\theta \Leftrightarrow C_x = C_y \Leftrightarrow f(C_x) = f(C_y)$ $\Leftrightarrow f(x) = f(y) \Leftrightarrow \Phi$ is kernel of homomorphism. □

3. Hyper Meet of two congruence relations

Definition 3.1: Let Θ_A be the collection of all congruence relations defined on hyperboolean algebra A. Then hyper meet of two congruence relations is denoted by $\theta_1 \otimes \theta_2$ and defined as $C_x (\theta_1 \otimes \theta_2) Cy \Rightarrow C_x\ \theta_1 C_y$ and $C_x\ \theta_2 C_y$.

Theorem3.2: Let θ_1 and θ_2 be any two congruence relations defined on an hyperboolean algebra A .Define a relation $\theta_1 \otimes \theta_2$ on A by $C_x (\theta_1 \otimes \theta_2) C_y \Leftrightarrow C_x\ \theta_1 C_y$ and $C_x\ \theta_2 C_y$. Then $\theta_1 \otimes \theta_2$ is a congruence relation defined on A such that $\theta_1 \otimes \theta_2$ is the fixed element of θ_1 & θ_2.

Proof: It is easy to prove that $C_x (\theta_1 \otimes \theta_2) C_x$ as $C_x \theta_1 C_x$ and $C_x \theta_2 C_x$. $C_x (\theta_1 \otimes \theta_2) C_y = C_y (\theta_1 \otimes \theta_2) Cy$ as $C_x \theta_1 C_y$ and $C_x \theta_2 C_y$. $\Rightarrow C_y\theta_1 C_x$ and $C_y\theta_2 C_x \Rightarrow C_x \theta_1 C_y$ and $C_x \theta_2 C_y$. now to prove transitivity, let $C_x (\theta_1 \otimes \theta_2) C_y$ and $C_y (\theta_1 \otimes \theta_2) C_z$, by definition , $C_x \theta_1 Cy$ and $C_x \theta_2 C_y$. $C_y \theta_1 C_z$ and $C_y \theta_2 C_z$. This implies $C_x \theta_1 C_z$ and $C_x \theta_2 C_z$. Therefore $C_x (\theta_1 \otimes \theta_2) C_z$. Let $f \in F$, let n be the corresponding integer, Let $C_{xi} (\theta_1 \otimes \theta_2) C_{yi}$ for all i This implies $C_{xi} \theta_1 C_{yi}$ and $C_{x i} \theta_2 C_{yi.}$ That is $f (C_{x 1}, C_{x 2} \ldots C_{x n}) \theta_1 f(C_{y1}, C_{y2},\ldots C_{yn})$ and $f(C_{x 1}, C_{x 2},\ldots C_{x n}) \theta_2 f(C_{y1}, C_{y2},\ldots C_{yn})$ that is $f (C_{x1}, C_{x2},\ldots C_{xn}) (\theta_1 \otimes \theta_2) f(C_{y1}, C_{y2},\ldots C_{yn})$. $\theta_1 \otimes \theta_2$ is a hypercongruence relation. To prove that $\theta_1 \otimes \theta_2$ is fixed element in hypersemilattice L. $\theta_1 \otimes \theta_2 \leq \theta_1$and $\theta_1 \otimes \theta_2 \leq \theta_2$ is obvious. That is $[\theta_1 \otimes \theta_2] \otimes \theta_1 = \{\theta_1 \otimes \theta_2 \}$ and $[\theta_1 \otimes \theta_2] \otimes \theta_2 = \{\theta_1 \otimes \theta_2 \}$Therefore by [1.4], $\theta_1 \otimes \theta_2$ is a fixed element of L. □

Definition3.3 : Con (L)denotes the set of all congruence relations on a hypersemilattice L. Then Con (L) forms complete Lattice with 0_L and 1_L, the fixed (smallest) and (absorbent) Largest lement of congruence relations.

Theorem 3.4: For hypersemilattice L with 0_L as fixed element and $\theta_1, \theta_2 \in$ Con (L), then there is a natural embedding of $L/ \theta_1 \otimes \theta_2 \rightarrow L/ \theta_1 \times L/ \theta_2$.

Proof: Let $\Psi = \theta_1 \otimes \theta_2$ .Then Φ / θ is a congruence relation on $L/ \theta_1 \otimes \theta_2$ and let $\Phi / \theta 1$,Φ / θ_2 be congruence relations on L/ θ_1 ,L/ θ_2 respectively. Define f: $L/ \theta_1 \otimes \theta_2 \rightarrow L/ \theta_1 \times L/ \theta_2$ by $f((C_x)_{\theta 1 \otimes \theta 2}) =\{((C_x)_{\theta 1} , (C_x)_{\theta 2}) / x \in L \}$ .Define a congruence relation Ψ by $(C_x), C_y) \in \Phi / \theta$ if and only $(x ,y) \in \Phi$. Let (C_x , C_y) and $(C_z\ C_w) \in \Phi/ \theta \Rightarrow (x, y) ,(z, w) \in \Phi$, This implies $(x \otimes z) ,(y \otimes w) \in \Phi/ \theta$ .That is $(C_{x\otimes z} , C_{y\otimes w})$ Hence Φ/ θ is congruence relation on L/ θ. Now let Φ/ θ is a congruence relation on L/ θ. Similarly we can prove Φ/ θ_1 and Φ/ θ_2 are congruence relations on L/ θ_1 and L/ θ_2. $f((C_x)_{\theta 1 \otimes \theta 2}) = f((C_y)_{\theta 1 \otimes \theta 2}) \Rightarrow ((C_x)_{\theta 1} , (C_x)_{\theta 2}) ==\{((C_y)_{\theta 1} , (C_y)_{\theta 2}) \Rightarrow (C_x)_{\theta 1} = (C_y)_{\theta 1}$ and $(C_x)_{\theta 2} =(C_y)_{\theta 2} \Rightarrow (C_x , C_y) \in \theta_1$ and $(C_x, C_y) \in \theta_2 \Rightarrow (C_x , C_y) \in \theta_1 \otimes \theta_2$ .But by Theorem [3.2],$\theta_1 \otimes \theta_2$ is a fixed element. That means by uniqueness property of fixed element of hypersemilattices $\theta_1 \otimes \theta_2 = 0_{L.}$ Therefore $C_x = Cy$. This implies f is one-one. To prove homomorphism Let $f((C_x \otimes C_y)_{\theta 1 \otimes \theta 2}) = [(Cx \otimes C_y)_{\theta 1} , C_x \otimes C_y)_{\theta 2}] = (C_x , C_x)_{\theta 1 \otimes \theta 2} \otimes (C_y , C_y)_{\theta 1 \otimes \theta 2} = (C_x)_{\theta 1 \otimes \theta 2} \otimes (C_y)_{\theta 1 \otimes \theta 2}$.Hence f is homomorphism. $L/ \theta_1 \otimes \theta_2 \cong L/ \theta_1 \times L/ \theta_{2.}$ □

Corollary 3.5: If hypersemilattice L has congruence relations θ_1 & θ_2 with $\theta_1 \otimes \theta_2 = 0_L$ then $L \rightarrow L/ \theta_1 \times L/ \theta_2$ (an embedding).

Proof: Let Φ be a congruence relation on L and Φ/θ_1, Φ/θ_2 be the congruence relations on L/θ_1 and L/θ_2 respectively. It can be easily checked using theorem [3.4]. Then define $\Psi : L \to L/\theta_1 \times L/\theta_2$ by $\Psi(x_\Phi) = \{((C_x)_{\theta 1}, (C_x)_{\theta 2}) / x \in L\}$ To prove Ψ is one-one let $\Psi(x_\Phi) = \Psi(y_\Phi) \Rightarrow ((C_x)_{\theta 1}, (C_x)_{\theta 2}) = ((C_y)_{\theta 1}, (C_y)_{\theta 2})$ .This implies $((C_x)_{\theta 1} = (C_y)_{\theta 1})$ and $((C_x)_{\theta 2} = (C_y)_{\theta 2})$, that is $(x,y) \in \theta_1$ and $(x,y) \in \theta_2$ therefore $(x,y) \in \theta_1 \otimes \theta_2$, but $\theta_1 \otimes \theta_2 = 0_L$ implies x=y. Let $\Psi(x_\Phi \otimes y_\Phi) = [(C_{x} \otimes C_y)_{\theta 1}, (C_x \otimes C_y)_{\theta 2}] = [(C_x)_{\theta 1} \otimes (C_y)_{\theta 1}] \otimes [(C_x)_{\theta 2} \otimes (C_y)_{\theta 2}] = [(C_x)_{\theta 1}, (C_x)_{\theta 2}] \otimes [(C_y)_{\theta 1}, (C_y)_{\theta 2}] = \Psi(x_\Phi) \otimes \Psi(y_\Phi)$. Ψ is a homomorphism. Therefore $A \to A/\theta_1 \times A/\theta_2$ is an embedding. □

4. Direct Product of Hypersemilattices

Definition 4.1: Let (L, ⊗) and (S, ⊗) be Hypersemilattices. Define binary operation on the Cartesian product Lx S as follows $(a_1,b_1) \bullet (a_2,b_2) = \{(c, d)/c \in a_1 \otimes a_2, d \in b_1 \otimes b_2\}$ for all $(a_1,b_1), (a_2,b_2) \in L \times S$,then (L x S, •) is called the direct product of Hypersemilattices (L, ⊗) and (S, ⊗).

Definition 4.2: Let $\{L_i / i \in I\}$ be a family of hypersemilattices. Then the direct product of Li, i∈I is the Cartesian product $\pi(L_i/i \in I) = \{(x_i), i \in I/x_i \in L_i\}$.

Theorem 4.3: The direct product of family of Hypersemilattices is again a hypersemilattice.

Proof: Let $\{L_i/ i \in I\}$ be a family of hypersemilattices. $L = \pi(L_i/i \in I) = \{(x_i), i \in I/x_i \in L_i\}$. Define hyperoperation ⊗ on L as follows:
$(x_i)_{i \in I} \otimes (y_i)_{i \in I} = \{(t_i)_{i \in I} / t_i \in x_i \otimes y_i)$. It is easy to observe that $(x_i)_{i \in I} \in (x_i)_{i \in I} \otimes (x_i)_{i \in I}$. $(x_i)_{i \in I} \otimes (y_i)_{i \in I} = (y_i)_{i \in I} \otimes (x_i)_{i \in I}$ and for any $x_i, y_i, z_i \in L_i$, $i \in I$, $((x_i)_{i \in I} \otimes (y_i)_{i \in I}) \otimes (z_i)_{i \in I} = \bigcup_{ti \epsilon xi \otimes yi} \{(p_i)_{i \in I}\, p_i \in t_i \otimes z_i\}$ and $(x_i)_{i \in I} \otimes ((y_i)_{i \in I} \otimes (z_i)_{i \in I}) = \bigcup_{qi \epsilon yi \otimes zi} \{(r_i)_{i \in I}\, r_i \in x_i \otimes q_i\}$.Let $(s_i)_{i \in I} \in (x_i)_{i \in I} \otimes ((y_i)_{i \in I} \otimes (z_i)_{i \in I})$. Then $(s_i)_{i \in I} \in \{(p_i)_{i \in I} / p_i \in t_i \otimes z_i\}$ for some $t_i \in x_i \otimes y_i \Rightarrow s_i \in t_i \otimes z_i$ for some $t_i \in x_i \otimes y_i \Rightarrow s_i \in (x_i \otimes y) \otimes z_i \Rightarrow s_i \in x_i \otimes (y \otimes z_i) \Rightarrow s_i \in x_i \otimes q_i$ for some $q_i \in y_i \otimes z_i \Rightarrow (s_i)_{i \in I} \in \{(r_i)_{i \in I}\, r_i \in x_i \otimes q_i\} \Rightarrow (s_i)_{i \in I} \in \bigcup_{qi \epsilon yi \otimes zi} \{(r_i)_{i \in I}\, r_i \in x_i \otimes q_i\} = (x_i)_{i \in I} \otimes ((y_i)_{i \in I} \otimes (z_i)_{i \in I})$ and hence $((x_i)_{i \in I} \otimes (y_i)_{i \in I}) \otimes (z_i)_{i \in I} \subseteq (x_i)_{i \in I} \otimes ((y_i)_{i \in I} \otimes (z_i)_{i \in I})$.Similarly we can prove $(x_i)_{i \in I} \otimes ((y_i)_{i \in I} \otimes (z_i)_{i \in I}) \subseteq ((x_i)_{i \in I} \otimes (y_i))_{i \in I}) \otimes (z_i)_{i \in I}$ .Hence the theorem. □

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