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The Fair Sharing Graph and its Helly Property

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Abstract

We define a new class of graphs called fair sharing graphs and prove that every three longest paths in a fair sharing graph share a common vertex. This verifies that the well-known conjecture that every three longest paths in a connected graph share a common vertex is true in this class of graphs.

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1 Introduction

In this study, we define a new class of graphs called fair sharing graphs. This is in connection with the question posed by Gallai in 1966 who asked whether every connected graph has a vertex that appears in all of its longest paths. Subsequent developments of the theory have shown that Gallai's conjecture is not true in general. H. Walter found the first counterexample of the conjecture and later Zamfirescu found a graph with 12 vertices in which there is no common vertex. However, if one asks whether every pair of longest paths in a connected graph share a common vertex the answer is surprisingly in the affirmative. This leads to the following conjecture:

Conjecture: For any three longest paths in a connected graph, there is a vertex which belongs to all three of them.

Maria Axenovich in her note entitled "*When do three longest paths have a common vertex?*" stated some developments in the problem and specifically proved that every three longest paths of a connected graph whose union form an outerplanar graph have a common vertex.

We use one of her two defined configurations (Q1) which is forbidden in the union of three longest paths of a connected graph to prove that every three longest paths in a fair sharing graph have a common vertex.

2 Definitions and Preliminaries

For a path $P = p_0, p_1, \ldots, p_k$, we denote a segment $p' = p_i, p_{i+1}, \ldots, p_j = p_i - p_j (j > i)$ of P as $(p_i - p_j)_P$. Moreover, we use the notation $I(p')$ for the set of interior vertices of p' and denote the length (number of edges) of a path P as $|P|$.

Definition 2.1 *A fair sharing graph is a connected graph G in which for every two longest paths of G say P and Q there is an end vertex p_0 and q_0 of P and Q respectively such that for every $v \in V(P) \cap V(Q)$,*

$$|(p_0 - v)_P| = |(q_0 - v)_Q| .$$

In this case, we say that p_0 and q_0 are the sharing references for P and Q. We will abbreviate this by saying that (p_0, q_0) is the sharing reference for (P, Q).

It is immediate from the definition that

$$|(\widehat{p_0} - v)_P| = |(\widehat{q_0} - v)_Q|$$

where $\widehat{p_0}$ and $\widehat{q_0}$ are the other end vertex of P and Q respectively. That is, $\widehat{p_0}$ and $\widehat{q_0}$ are also sharing references for P and Q. Throughout our discussion, if p_0 is an end vertex of a path P then $\widehat{p_0}$ denotes the other end vertex of P. We can see further that for all $p_i, p_j \in V(P)$ we have

$$|(p - p_i)_P| = |(p - p_j)_P| \Leftrightarrow p_i = p_j.$$

In the definitions below, A, B and C denote any three longest paths of a connected graph.

Definition 2.2 ***Configuration C1*** *A cycle which is a union of segments s_0 and s_1 of A and B respectively such that $I(s_0) \cap V(C) \neq \phi$, interior of s_1 does not contain any vertex from A and C and end vertices of s_0 and s_1 are not in C for some longest paths A, B and C.*

Configuration C2 *A cycle which is a union of internally disjoint segments s_0, s_1 and s_2 of longest paths A, B and C respectively such that the interior vertices of s_0 and s_2 do not contain any vertex from B and the interior vertices of s_1 and s_2 do not contain any vertex from A.*

Note that Configuration $C2$ is the Configuration $Q1$ defined by Maria Axenovich which is forbidden in the union of any three longest paths.

3 Main Result

The main aim of this paper is to prove that in a fair sharing graph every three longest paths in it have a common vertex. To achieve this, we divide it in two steps. First, we need to establish the following:

Theorem 3.1 *Let P, Q and R be any three longest paths of a fair sharing graph G with (p_0, q_0), (r_0, q_0) and (p_0, r_0) as the sharing reference for (P, Q), (R, Q) and (P, R) respectively. Then P, Q and R share a common vertex.*

We achieve this through a series of lemma.

Lemma 3.2 *Let P, Q and R be three longest paths of a fair sharing graph. Let $v_1 \in V(P) \cap V(R)$, $v_2 \in V(Q) \cap V(R)$ and $v_1, v_2 \notin V(P) \cap V(Q) \cap V(R)$. Suppose that (p_0, q_0) is the sharing reference for (P, Q), (r_0, q_0) for (R, Q) and (p_0, r_0) for (P, R) then $|(p_0 - v_1)_P| \neq |(q_0 - v_2)_Q|$.*

Proof: Suppose $|(p_0 - v_1)_P| = |(q_0 - v_2)_Q|$ then $|(r_0 - v_1)_R| = |(q_0 - v_2)_Q| = |(r_0 - v_2)_R|$. This implies that $v_1 = v_2$ and hence $v_1 \in V(P) \cap V(Q) \cap V(R)$ a contradiction.

Lemma 3.3 *Let P and Q be two longest paths of a fair sharing graph with (p_0, q_0) the sharing reference of (P, Q). If $p \in V(P)$ and $q \in V(Q)$ such that $|(p_0 - p)_P| < |(q_0 - q)_Q|$ then $(p_0 - p)_P$ and $(q - \widehat{q_0})_Q$ are*

Proof: Suppose there exists a $v \in V((p_0 - p)_P \cap (q - \widehat{q_0})_Q)$ then $|(p_0 - v)_P| < |(q_0 - q)_Q| \leq |(q_0 - v)_Q|$. A contradiction.

Lemma 3.4 *Let P and Q be any two longest paths of a fair sharing graph with (p_0, q_0) as the sharing reference of (P, Q) and let $V(P) \cap V(Q) = \{v_1, v_2, \ldots, v_m\}$ then $|(v_i - v_j)_P| = |(v_i - v_j)_Q|$.*

Proof: If we suppose that $|(v_i - v_j)|_P \neq |(v_i - v_j)|_Q$ then $|P| \neq |Q|$.

Lemma 3.5 *Let P and Q be two longest paths paths of a fair sharing graph G with (p_0, q_0) the sharing reference for (P, Q) and let $V(P) \cap V(Q) = \{v_1, v_2, \ldots, v_m\}$ then $(p_0 - v_i)_P \cup (v_i - \hat{q_0})_Q)$ and $(q_0 - v_i)_Q \cup (v_i - \hat{p_0})_P)$ are also longest paths in G for any i.*

Proof: Let $x \in V((p_0 - v_i)_P)$, $x \neq v_i$ and $y \in V((v_i - \widehat{q_0})_Q)$ then

$$|(p_0 - x)_P| < |(p_0 - v_i)_P| = |(q_0 - v_i)_Q| .$$

By Lemma 3.3, $(p_0 - x)_P$ and $(v_i - \widehat{q_0})_Q$ are disjoint. Hence, $(p_0 - v_i)_P \cup (v_i - \widehat{q_0})_Q$ is a path with length $\left|(p_0 - v_i)_P \cup (v_i - \widehat{q_0})_Q\right| = |(p_0 - v_i)_P \cup (v_i - \widehat{p_0})_P| = |P|$. Similarly, $(q_0 - v_i)_Q \cup (v_i - \widehat{p_0})_P)$ is also a longest path in G.

Lemma 3.6 *Let A, B and C be any three longest paths of a fair sharing graph G with (a_0, b_0), (c_0, b_0) and (a_0, c_0) the sharing references for (A, B), (C, B) and (A, C), respectively. Then the union of A, B and C contains neither $C1$ nor $C2$.*

Proof: It suffices to show that the union of A, B and C does not contain $C1$. Suppose the union of A, B and C contains $C1$. Let s_0 and s_1 be the segments of A and B, respectively that satisfy Configuration $C1$. Let $s_1 = x_0, x_1, \ldots, x_m$ such that the higher the subscript of x the farther from the sharing reference. Clearly, the end vertices of s_0 are also x_0 and x_m. Furthermore, for all $b \in V(B)$ such that

$$|(b_0 - x_0)_B| < |(b_0 - b)_B| < |(b_0 - x_m)_B| \Longrightarrow b \notin V(A) \cup V(B).$$

Let $v_n \in I(s_0) \cap V(C)$ such that for all $v \in I(s_0) \cap V(C)$ we have $|(a_0 - v)_A| \leq |(a_0 - v_n)_A|$. It follows that $V((v_n - x_m)_A \cap V(C)) = \{v_n\}$ and $|(v_n - x_m)_A| \geq 1$. Observe that for each $s \in I(s_0)$ we have

$$|(a_0 - x_0)_A| = |(b_0 - x_0)_B| < |(a_0 - s)_A| < |(b_0 - x_m)_B| = |(a_0 - x_m)_A| .$$

By Lemma 3.3, $s \notin V(B)$ and $V((v_n - x_m)_A \cap V(B)) = \{x_m\}$. Moreover, we have $H = s_1 \cup (v_n - x_m)_A \cup (v_n - \widehat{c_0})_C$ a path in G. Since for all $a \in V((a_0 - x_0)_A)$, $|(a_0 - a)_A| < |(c_0 - v_n)_C|$ then $V((a_0 - x_0)_A)$ and $V((v_n - x_m)_A) \cup (v_n - \widehat{c_0})$ are disjoint. Thus, $(a_0 - x_0)_A \cup H$ is again a path in G with length greater than $|A|$. This completes the proof.

Proof of Theorem 3.1

Let P, Q and R be any three longest paths of a fair sharing graph G. Since G is a connected graph then R must have a vertex in common with P and

Q. Suppose that P, Q and R do not have a common vertex. Then there exist $x \in V(P) \cap V(R)$ and $y \in V(Q) \cap V(R)$ such that $I((x-y)_R)$ does not contain any vertex from P or Q with $|(x-y)_R| \geq 1$. Moreover, by Lemma 3.2 $|(r_0 - x)_R| \neq |(r_0 - y)_R|$. Without loss of generality, let $|(r_0 - x)_R| < |(r_0 - y)_R|$. Suppose there exists $v \in V(P) \cap V(Q)$ such that

$$|(p_0 - x)_P| < |(p_0 - v)_P| = |(q_0 - v)_Q| < |(q_0 - y)_Q| .$$

By Lemma 3.5 $F = (p_0 - v)_P \cup (v_0 - \widehat{q_0})_Q$ is path of maximum length in G with $V(F) \cap V(R) = \{x, y\}$. Since the interior of the segment $(x-y)_R$ does not contain any vertex from P or Q then it also does not contain any vertex from F. Again, by Lemma 3.5 $F' = (q_0 - v)_Q \cup (v - \widehat{p_0})_P$ is also a longest path in G and $V((x-y)_F) \cap V(F') \neq \phi$. Thus, with R as B, F as A, F' as C, $(x-y)_R$ as s_1 and $(x-y)_F$ as s_0 we have Configuration $C1$. On the other hand, suppose there does not exist $v \in V(P) \cap V(Q)$ such that

$$|(p_0 - x)_P| < |(p_0 - v)_P| = |(q_0 - v)_Q| < |(q_0 - y)_Q| .$$

There are two possible cases: (i) there exists $v \in V(P) \cap V(Q)$ such that

$$|(p_0 - x)_P| < |(q_0 - y)_Q| < |(p_0 - v)_P| = |(q_0 - v)_Q|$$

or (ii) there exists $v \in V(P) \cap V(Q)$ such that

$$|(p_0 - v)_P| = |(q_0 - v)_Q| < |(p_0 - x)_P| < |(q_0 - y)_Q| .$$

Considering (i), take $v' \in V(P) \cap V(Q)$ such that for all $v \in V(P) \cap V(Q)$ with $|(q_0 - y)_Q| < |(p_0 - v)_P|$ we have $|(p_0 - v')_P| < |(p_0 - v)_P|$. Then $I((x - v')_P)$ does not contain any vertex from Q and $I((y - v')_Q)$ does not contain any vertex from P. Observe that $(x - v')_P \cup (v' - y)_Q \cup (x - y)_R$ is a cycle with $|(x - v')_P| \geq 1$ and $|(y - v')_Q| \geq 1$. Thus, we have Configuration $C2$. By a similar argument (ii) will follow.

In Theorem 3.1 if (p_0, q_0) and (r_0, q_0) are the sharing references for (P, Q) and (R, Q) respectively then it may be possible that $(p_0, \widehat{r_0})$ is the sharing reference for (P, R) instead of (p_0, r_0). If we can show that (p_0, r_0) is always a sharing reference for (P, R) then we are able to show that every three longest paths in a fair sharing graph share a common vertex.

We need the following lemma.

Lemma 3.7 *Let P and Q be any two longest paths in a fair sharing graph G. Then the following hold:*

(i) If $|V(P) \cap V(Q)| \geq 2$ then the sharing references for (P, Q) are (p_0, q_0) and $(\widehat{p_0}, \widehat{q_0})$.

(ii) If $|V(P) \cap V(Q)| = 1$ then the sharing references for (P, Q) are (p_0, q_0), $(\widehat{p_0}, \widehat{q_0})$, $(\widehat{p_0}, q_0)$ and $(p_0, \widehat{q_0})$.

Proof: (i) Suppose $|V(P) \cap V(Q)| \geq 2$. For simplicity, we may assume $V(P) \cap V(Q) = \{v_1, v_2\}$. We may assume further that v_1 is closer to p_0 than v_2. Suppose that $(p_0, \widehat{q_0})$ is a sharing reference for (P, Q). Then $|(p_0 - v_1)_P| = |(v_1 - \widehat{q_0})_Q|$. It follows that

$$|(p_0 - v_2)_P| > |(p_0 - v_1)_P| = |(v_1 - \widehat{q_0})_Q| > |(v_2 - \widehat{q_0})_Q| .$$

A similar reasoning applies for $(\widehat{p_0}, q_0)$.

(ii) The second statement is trivial.

Theorem 3.8 *Let P, Q and R be any three longest paths of a fair sharing graph G. If (p_0, q_0) and (r_0, q_0) are the sharing references for (P, Q) and (R, Q) respectively then (p_0, r_0) is a sharing reference for (P, R).*

Proof: Clearly, this follows from Lemma 3.7. This completes the proof.

Combining Theorem 3.1 and Theorem 3.8 we now have the main theorem.

Theorem 3.9 *Let G be a fair sharing graph and let P, Q and R be any three of its longest paths. Then there there is a $v \in V(G)$ such that $v \in V(P) \cap V(Q) \cap V(R)$.*

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References

[1] T. Zamfirescu, *Intersecting longest paths or cycles: a short survey.* An. Univ. Craiova Ser. Mat. Inform. 28(2001), 1-9.

[2] M. Axenovich, *When do three longest paths have a common vertex?*, Discrete Mathematics, Algorithms and Applications, , 1:1(2009), 115-120.

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