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On h-Hemiregular Hemirings

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Abstract. In this paper the notion of a weakly left(right) h-hemiregular hemiring is introduced and characterizations of it are given. It is also proved that every weakly h-hemiregular hemiring is h-hemiregular and the converse is also true if the set of h-bi-ideals and h-quasi-ideals coincides. Some results on prime and semiprime bi-ideals of hemirings are also presented.

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1. INTRODUCTION AND PRELIMINARIES

J. Zhan and W. A. Dudek [4] have introduced the concept of h-hemiregular hemirings and they characterized prime fuzzy h-ideals of h-hemiregular hemirings by fuzzy h-ideals. Also, La Torre[2] studied the properties of h-ideals and $k-$ideals of hemirings and proved some theorems analogous to ring theory. A semiring is a non-empty set S together with two binary operations called addition "+" and multiplication "." (denoted by juxtaposition) such that $(S, +)$ and $(S, .)$ are semigroups and both are connected by ring-like distributivity. A semiring S with zero means that there exists an element $0 \in S$ such that $0.x = x.0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$. A semiring with zero and a commutative semigroup $(S, +)$ is called hemiring. A hemiring S is said to be commutative if $(S, .)$ is commutative. Throughout this paper S will denote a hemiring. A non-empty subset A of S is called left(right) ideal of S if A

is closed with respect to addition and $SA \subseteq A(AS \subseteq A)$. A subset A of a hemiring S is called an ideal if it is both a left and a right ideal of S. A left(right) ideal A of S is called left(right)h-ideal of S if $a, b \in A$, $x, z \in S$ and $x + a + z = b + z \in S$ implies $x \in A$. The h-closure $\overline{A}$ of A in a hemiring S is defined as $\overline{A} = \{x \in S | x + a + z = b + z \text{ for some } a, b \in A, z \in S\}$. It is easy to see that if A is a left(resp. right) ideal of S, then $\overline{A}$ is the smallest right(resp. left) h-ideal of S containing A. A subset Q of a hemiring S is called an h-quasi-ideal of S if Q is closed under addition, $\overline{SQ} \cap \overline{QS} \subseteq Q$ and $\overline{Q} = Q$. A subset B of a hemiring S is called a h-bi-ideal of S if B is closed under addition, $\overline{BSB} \subseteq B$ and $\overline{B} = B$. A hemiring S is said to be h-hemiregular if for each $a \in S$, there exist $x, y, z \in S$ such that $a + axa + z = aya + z$.

2. PRIME AND SEMIPRIME h-BI-IDEALS

In this section we define prime and semiprime h-bi-ideals in hemirings. Before proving the results we require the following useful lemma.

Lemma 2.1 (Lemma 3.3[4]). *Let S be a hemiring and $A, B \subseteq S$, then $\overline{AB} = \overline{\overline{A}\,\overline{B}}$.*

Definition 2.2. An h-bi-ideal B of a hemiring S is called prime if $\overline{xSy} \subseteq B$implies $x \in B$ or $y \in B$ and an h-bi-ideal of the hemiring S is called semiprime if $\overline{xSx} \subseteq B$implies $x \in B$.

Example 2.3. *Let N be the set of all positive integers. Then N is a semiring under the usual addition and multiplication. Let $B = 2N$. Thus $BNB = 4N \subseteq 2N = B$. Hence B is a h-bi-ideal and a prime h-bi-ideal of N.*

Proposition 2.4. *An h-bi-ideal of a hemiring S is prime if and only if $\overline{RL} \subseteq B$ for each right h-ideal R of S and left h-ideal L of S implies $R \subseteq B$ or $L \subseteq B$.*

Proof. Suppose that B is a prime h-bi-ideal of S and $\overline{RL} \subseteq B$. Suppose $R \nsubseteq B$. For each $r \in R \setminus B$, and $l \in L$, we have $\overline{rSl} \subseteq \overline{RL} \subseteq B$. Since B is prime and $r \notin B$, therefore for all $l \in L$, we have $l \in B$ proving $L \subseteq B$. Conversely, suppose that $\overline{RL} \subseteq B$ implies $R \subseteq B$ or $L \subseteq B$ for any right h-ideal R and left h-ideal L of S. Let $x, y \in S$ such that $\overline{xSy} \subseteq B$. Now $\overline{\overline{(xS)}\,\overline{(Sy)}} \subseteq \overline{xSy} \subseteq B$. Since $\overline{xS}$ is a right h-ideal and $\overline{Sy}$ is a left h-ideal of S, we have

$\overline{xS} \subseteq B$ or $\overline{Sy} \subseteq B$. Suppose $\overline{xS} \subseteq B$. Clearly, $\overline{xS + Z_0x}$ and $\overline{Sx + Z_0x}$ are respectively the right h-ideal and the left h-ideal of S generated by x. Let $z \in \overline{\overline{xS + Z_0x}\ \overline{Sx + Z_0x}} = \overline{xS + Z_0xSx + Z_0x} \subseteq \overline{Z_0xSx + Z_0x^2}$. Therefore $z + \sum(m_ixs_ix + n_ix^2) + h = \sum(m_i'xs_i'x + n_i'x^2) + h$ for $m_i, n_i, m_i', n_i' \in Z_0$ and $s_i, s_i', h \in S$. Since $xS \subseteq \overline{xS} \subseteq B$, therefore x^2, xs_ix and $xs_i'x \in B$. Thus $z \in \overline{B} = B$. Hence $\overline{\overline{xS + Z_0x}\ \overline{Sx + Z_0x}} \subseteq B$. Therefore by hypothesis, we have $x \in xS + Z_0x \subseteq \overline{xS + Z_0x} \subseteq B$. Similarly, if $\overline{Sy} \subseteq B$, then $y \subseteq B$. Hence B is prime ideal of S. □

Theorem 2.5. *A prime h-bi-ideal of a hemiring S is a prime one sided h-ideal of S.*

Proof. Let B be a prime h- bi-ideal of a hemiring S. Since $\overline{\overline{BS}\ \overline{SB}} \subseteq \overline{BSB} \subseteq B$ and $\overline{BS}$ and $\overline{SB}$ are one sided h-ideals of S, therefore by the above theorem, we have $\overline{BS} \subseteq B$ or $\overline{SB} \subseteq B$. Thus B is a one-sided h-ideal of S. □

Theorem 2.6. *Let B be a semiprime h-bi-ideal of a hemiring S. Then B is a h-quasi-ideal of S*

Proof. Suppose $a \in \overline{(BS)} \cap \overline{(SB)}$. Then $\overline{aSa} \subseteq \overline{\overline{(BS)}\ S\ \overline{(SB)}} \subseteq \overline{BSB} \subseteq B$. Since B is semiprime, we have $a \in B$. Therefore $\overline{(BS)} \cap \overline{(SB)} \subseteq B$ and thus B is a h- quasi-ideal of S. □

Let B be an h-bi-ideal of a hemiring S. Then define

$$L_B = \{x \in B : \overline{Sx} \subseteq B\} \qquad R_B = \{x \in B : \overline{xS} \subseteq B\}$$

$$I_L = \{y \in L_B : \overline{yS} \subseteq L_B\} \qquad I_R = \{y \in R_B : \overline{Sy} \subseteq R_B\}$$

Proposition 2.7. *Let B be an h-bi-ideal of a hemiring S. Then L_B (resp.R_B) is a left (resp. right) h-ideal of S contained in B if L_B (resp. R_B) is non-empty.*

Proof. Let $x \in L_B$ and $s \in S$. Then $sx \in Sx \subseteq \overline{Sx} \subseteq B$. Now $\overline{Ssx} \subseteq \overline{S\overline{Sx}} = \overline{SSx} \subseteq \overline{Sx} \subseteq B$. Thus we have $sx \in L_B$. Consequently $\overline{SL_B} \subseteq L_B$. So L_B is a left ideal as well as a left h-ideal of S (as B is a h-bi-ideal of S). Similarly we can prove that R_B is a right h- ideal of S. □

Proposition 2.8. *Let B be an h-bi-ideal of a hemiring S. If I_L (resp. I_R) is non-empty then I_L (resp I_R) is the largest h-ideal of S contained in B. Moreover $I_L = I_R$.*

Proof. Let $x \in I_L$. Then $I_L \subseteq L_B \subseteq B$ implies $x \in L_B$ and $x \in B$. That is $\overline{Sx} \subseteq B$. Then $\overline{Ssx} \subseteq \overline{Sx} \subseteq B$ for each $s \in S$. This implies $sx \in L_B$. Since L_B is a left h-ideal of S (by above Proposition) and $\overline{xS} \subseteq L_B$ therefore $\overline{sxS} \subseteq \overline{SL_B} \subseteq L_B$. Thus $sx \in I_L$. That is $\overline{SI_L} \subseteq I_L$. Hence I_L is a left h-ideal of S. Similarly, we can show that I_L is a right h-ideal of S. Hence I_L is an h-ideal of S contained in B.

Let I be any h-ideal of S contained in B. Then $\overline{SI} \subseteq I \subseteq B$. This implies $I \subseteq L_B$. Now $\overline{IS} \subseteq I \subseteq L_B$. This implies $I \subseteq I_L$. Hence I_L is the largest h-ideal of S contained in B. Similarly we can prove that I_R of S contained in B. Since I_L, I_R are the largest h-ideals of S contained in B, therefore $I_L = I_R$. □

Define $I_B = I_L = I_R$.

Proposition 2.9. *Let B be a prime h-bi-ideal of a hemiring S. Then I_B is a prime h-ideal of S.*

Proof. Let B be a prime h-bi-ideal of S. Suppose $\overline{RL} \subseteq I_B$ for any right h-ideal R and left h-ideal L of S. Now $I_B \subseteq L_B \subseteq B$ implies $\overline{RL} \subseteq B$. Since B is prime, therefore $R \subseteq B$ or $L \subseteq B$ (by Theorem 2.4). Also I_B is the largest h-ideal contained in B, therefore $R \subseteq I_B$ or $L \subseteq I_B$. Hence I_B is a prime ideal of S. □

Theorem 2.10. *A hemiring S is h-hemiregular if and only if every h-bi-ideal of S is semiprime.*

Proof. Let S be an h-hemiregular hemiring and B be an h-bi-ideal of S. Suppose $\overline{aSa} \subseteq B$ for $a \in S$. Since S is h-hemiregular there exist $x, y, h \in S$ such that $a + axa + h = aya + h$. Since $axa, aya \in aSa$, therefore $a \in \overline{aSa} \subseteq B$ and thus B is semiprime. Conversely, assume that every h-bi-ideal of S is semiprime. Since for $a \in S$, $\overline{aSa}$ is h-bi-ideal of S because $\overline{\overline{aSa}S\overline{aSa}} \subseteq \overline{\overline{aSa}\ \overline{S}\ \overline{aSa}} \subseteq \overline{aSa}$. Moreover, $\overline{aSa}$ is semiprime, therefore $a \in \overline{aSa}$. Hence there exist $x, y, h \in S$ such that $a + axa + h = aya + h$ and therefore S is h-hemiregular. □

A hemiring S is B-simple if it has no nonzero proper h-bi-ideal of S.

Proposition 2.11. *Let S be a hemiring. Then S is h-hemiregular and $B-$simple if and only if $\overline{aSa} = S$ for any $0 \neq a \in S$.*

Proof. Let S be a $B-$simple h- hemiregular hemiring. Then for any $0 \neq a \in S$, there exist $x, y, z \in S$ such that $a + axa + z = aya + z$. That is, $a \in \overline{aSa}$ which implies that $\overline{aSa} \neq 0$. Since $\overline{aSa}$ is a h-bi-ideal of S and also S is $B-$simple, we have $S = \overline{aSa}$ for any $0 \neq a \in S$. Conversely, let $0 \neq a \in S$. Then by hypothesis $S = \overline{aSa}$, which yields that S is h- hemiregular. Let B be a non zero h-bi-ideal of S and let $0 \neq b \in S$. Then $S = \overline{bSb} \subseteq \overline{BSB} \subseteq B$. Thus $B = S$ and S is $B-$simple. □

Proposition 2.12. *If B is a h-bi-ideal of S, and A is a h-ideal of S, then if A is $B-$simple such that $A \cap B \neq \emptyset$, then $A \subseteq B$.*

Proof. Suppose that A is $B-$simple such that $A \cap B \neq \emptyset$ and $a \in A \cap B$. It is easy to show that $\overline{aAa}$ is h-bi-ideal of A. Since A is $B-$ simple therefore $\overline{aAa} = A$. Hence $A = \overline{aAa} \subseteq \overline{BSB} \subseteq B$. Hence $A \subseteq B$. □

3. WEAKLY h-HEMIREGULAR HEMIRINGS

In this section, we define weakly h-hemiregular hemirings and give some characterizations related to the same.

Definition 3.1. A hemiring S is said to be weakly right h-hemiregular if for each $a \in S$, there exist $x_1, x_2, x_3, x_4, z \in S$ such that $a + ax_1ax_2 + z = ax_3ax_4 + z$ and said to be weakly left h-hemiregular if $a + x_1ax_2a + z = x_3ax_4a + z$.

In case of a commutative hemiring, a weakly left(right)h-hemiregular hemiring is h-hemiregular. The following theorem shows that both concepts also coincide, when the sets of h-bi-ideals and h-quasi-ideals coincide.

Theorem 3.2. *Let S be a hemiring such that the set of all h-bi-ideals coincides with the sets of all h-quasi-ideals. Then S is h-hemiregular if and only if it is weakly h-hemiregular.*

Proof. Suppose S is h-hemiregular, then for every $a \in S$ there exist $x, y, h \in S$ such that

$$a + axa + h = aya + h. \tag{1}$$

This implies

$$axa + axaxa + axh = axaya + axh \tag{2}$$

and

(3) $$axa + axaxa + hxa = ayaxa + hxa.$$

From these two equations, we have

(4) $$ayaxa + u = axaya + u \quad \text{for some } u \in S.$$

Again, from(1) we have

(5) $$aya + axaya + hya = ayaya + hya.$$

Adding (2)and (5) and then $a+h$ on both sides, we get $a + aya + ayaxa + axaya + axh + hya + h = a + axa + axaxa + axh + ayaya + hya + h$. This implies $a + ayaxa + axaya + v = axaxa + ayaya + v$ for some $v \in S$. Adding $axaya + ayaxa + u + u$ on both sides and using (4), we get $a + a(x + x + x + x)a(ya) + u + u + v = a(x+y)a((x+y)a) + u + u + v$ or $a + az_1az_2 + z = az_3az_4 + z$ for some $z, z_1, z_2, z_3, z_4 \in S$. Similarly, it can be shown that S is weakly left h-hemiregular. Therefore S is weakly h-hemiregular. Conversely, suppose that $a \in S$ is weakly h-hemiregular i.e. $a \in \overline{aSaS}$ and $a \in \overline{SaSa}$. Since $\overline{aSa}$ is a h-bi-ideal of S (by Theorem 2.6)and every h-bi-ideal of S is a h-quasi-ideal of S, therefore $a \in \overline{\overline{aSaS}} \cap \overline{S\overline{aSa}} \subseteq \overline{aSa}$. Therefore there exist $x, y, z \in S$ such that $a + axa + z = aya + z$ and therefore S is h-hemiregular. □

Theorem 3.3. *A hemiring S is weakly right h-hemiregular if and only if for all right h-ideals R_1, R_2, we have $R_1 \cap R_2 \subseteq \overline{R_1R_2}$.*

Proof. Let S be weakly right h-hemiregular and let $a \in R_1 \cap R_2$. Then there exist $x_1, x_2, x_3, x_4, z \in S$ such that $a + ax_1ax_2 + z = ax_3ax_4 + z$. As $ax_1ax_2, ax_3ax_4 \in R_1SR_2S \subseteq R_1R_2$, therefore $a \in \overline{R_1R_2}$. Thus $R_1 \cap R_2 \subseteq \overline{R_1R_2}$.

Conversely, let $a \in S$. The principal right ideal of S generated by a is given by $aS + Z_0a$, where $Z_0=\{0, 1, 2, . . \}$. Now,

$$\begin{aligned} aS + Z_0a &\subseteq \overline{aS + Z_0a} = \overline{aS + Z_0a} \cap S \\ &\subseteq \overline{\overline{(aS + Z_0a)}\ \overline{S}} = \overline{(aS + Z_0a)S} = \overline{aS} \end{aligned}$$

as S is trivially a right h-ideal.

Again, $a = a.0 + 1.a \in aS + Z_0a \subseteq \overline{aS}$. Now, $a \in \overline{aS} = \overline{aS} \cap \overline{aS} \subseteq \overline{\overline{aS}\ \overline{aS}} = \overline{aSaS}$, because $\overline{aS}$ is a right h-ideal of S. Therefore there exist $x_1, x_2, x_3, x_4, z \in$

S such that $a+ax_1ax_2+z = ax_3ax_4+z$. Hence S is weakly right h-hemiregular. □

Theorem 3.4. *A hemiring S is weakly right h-hemiregular if and only if for all right h-ideal R we have $R = \overline{R^2}$*

Proof. Suppose S is weakly right h-hemiregular. Then putting $R_1 = R_2 = R$ in above theorem, we get $R \subseteq \overline{R^2}$. Also, since $R^2 \subseteq R$, we have $\overline{R^2} \subseteq \overline{R} = R$. Thus $\overline{R^2} = R$.

Conversely, let R_1, R_2 be two right h-ideals of S. Then $R_1 \cap R_2 = \overline{(R_1 \cap R_2)^2} \subseteq \overline{R_1R_2}$,because $a \in \overline{(R_1 \cap R_2)^2}$ implies $a+r_1r_2+z = r_3r_4+z$ where $r_1, r_2, r_3, r_4 \in R_1 \cap R_2$ and $z \in S$, which implies $a \in \overline{R_1R_2}$. Hence by the above theorem, S is weakly right h-hemiregular. □

Similarly, we can prove the following

Theorem 3.5. *A hemiring S is weakly left h-hemiregular if and only if for all left h-ideals L_1, L_2 we have $L_1 \cap L_2 \subseteq \overline{L_1L_2}$*

Theorem 3.6. *A hemiring S is weakly right h-hemiregular if and only if for all right h-ideal R and h-ideal I we have $R \cap I \subseteq \overline{RI}$.*

Proof. The result can easily be proved by Theorem 2.5 and Lemma 2.1. □

Theorem 3.7. *let S be a hemiring and I be an ideal of S. Then I is hemiregular if and only if there exist $a \in I$ such that $a \in \overline{aIaI}$.*

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