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Bi-ideals in Ordered Γ-Semigroups

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Abstract

In this note, we show in the line of [3] that an ordered Γ-semigroup is left and right simple if and only if it does not contain proper bi-ideals.

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An *ordered semigroup* [1] is a semigroup $(S, \cdot)$ together with a partial order $\leq$ on S that is compatible with the semigroup operation, meaning that $x \leq y$ implies $z \cdot x \leq z \cdot y$ and $x \cdot z \leq y \cdot z$ for all x, y, z in S. It is customary to write $(S, \cdot, \leq)$ rather than $(S, \cdot)$. Let $(S, \cdot, \leq)$ be an ordered semigroup. A nonempty subset A of S is called a *left (respectively, right, bi-) ideal* of S if (i) $SA \subseteq A$ (respectively, $AS \subseteq A$, $ASA \subseteq A$) and (ii) If $x \in A$ and $y \in S$ such that $y \leq x$, then $y \in A$. S is said to be *left (respectively, right) simple* if S does not contain proper left (respectively, right) ideals. In [3], the authors proved that S is left and right simple if and only if S does not contain proper bi-ideals. The purpose of this note is to show that the result is also valid on ordered Γ-semigroups.

The concept of Γ-semigroup has been introduced by Sen in [4]. Thereafter, the definition defined by Sen was changed by Sen and Saha [5] as follow: Let S and Γ be two non-empty sets. Then S is called a Γ-semigroup if

(1) $x\alpha y \in S$ and

(2) $(x\alpha y)\beta z = x\alpha(y\beta z)$

for all $x, y, z \in S$ and all $\alpha, \beta \in \Gamma$.

By adding the uniqueness condition to the definition defined by Sen and Saha, Kehayopulu defined a Γ-semigroup in [2]:

Definition 1. Let S and Γ be two non-empty sets. Then S is called a Γ-semigroup if

(1) $x\alpha y \in S$ for all $x, y \in S$ and all $\alpha \in \Gamma$.

(2) If $x, y, z, w \in S$ and $\alpha, \beta \in \Gamma$ such that $x = z, y = w$ and $\alpha = \beta$, then $x\alpha y = z\beta w$.

(3) $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in S$ and all $\alpha, \beta \in \Gamma$.

In this paper, we follow Definition 1. Let S be a Γ-semigroup. For $A, B \subseteq S$, let

$$A\Gamma B = \{a\alpha b : a \in A, b \in B, \alpha \in \Gamma\}.$$

If $x \in S$, let $A\Gamma x = A\Gamma\{x\}$ and $x\Gamma A = \{x\}\Gamma A$.

A Γ-semigroup (S, Γ) is called an *ordered* Γ*-semigroup* [6] if there is a partial order $\leq$ on S such that

$$x \leq y \text{ implies } x\alpha z \leq y\alpha z \text{ and } z\alpha x \leq z\alpha y$$

for any $x, y, z \in S$ and all $\alpha \in \Gamma$.

Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. For $A \subseteq S$, let

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

Let $A, B \subseteq S$. The following are well-known: $A \subseteq (A]$, $A \subseteq B$ implies $(A] \subseteq (B]$, $(A]\Gamma(B] \subseteq (A\Gamma B]$.

Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. A nonempty subset A of S is called a *left (respectively, right) ideal* of S if the following hold.

(i) $S\Gamma A \subseteq A$ (respectively, $A\Gamma S \subseteq A$).

(ii) If $x \in A$ and $y \in S$ such that $y \leq x$, then $y \in A$.

If A is both a left and a right ideal of S, then A is called an *ideal* of S.

Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. A nonempty subset A of S is called a *bi-ideal* of S if the following hold.

(i) $A\Gamma S\Gamma A \subseteq A$.

(ii) If $x \in A$ and $y \in S$ such that $y \leq x$, then $y \in A$.

A left (respectively, right, bi-) ideal A of an ordered Γ-semigroup $(S, \Gamma, \leq)$ is said to be *proper* if $A \neq S$. S is said to be *left (respectively, right) simple* if it does not contain proper left (respectively, right) ideals.

Note that every left and right ideal of an ordered Γ-semigroup S is a bi-ideal of S. Indeed: if A is a left (right) ideal of S, then $A\Gamma S\Gamma A \subseteq A\Gamma A \subseteq S\Gamma A \subseteq A$ ($A\Gamma S\Gamma A \subseteq A\Gamma A \subseteq A\Gamma S \subseteq A$).

A bi-ideal A of an ordered Γ-semigroup $(S, \Gamma, \leq)$ is called a *subidempotent bi-ideal* if $A\Gamma A \subseteq A$.

An ordered Γ-semigroup $(S, \Gamma, \leq)$ is said to be *regular* if for any $a \in S$, $a \in (a\Gamma S\Gamma a]$.

Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. As in [3], we have the following.

(1) $(S, \Gamma, \leq)$ is left (respectively, right) simple if and only if $(S\Gamma x] = S$ (respectively, $(x\Gamma S] = S$) for all $x \in S$.

Indeed: assume that $(S\Gamma x] = S$ for all $x \in S$. Let A be a left ideal of S and $a \in A$. We have $(S\Gamma a] = S$. If $y \in S$, then $y \leq z\alpha a$ for some $z \in S$ and for some $\alpha \in \Gamma$. Since $z\alpha a \in A$, $y \in A$. Hence $A = S$. The opposite direction follows by $(S\Gamma x]$ is a left ideal of S for all $x \in S$. Similarly, S is right simple if and only if $(x\Gamma S] = S$ for all $x \in S$.

(2) If $(S, \Gamma, \leq)$ is left and right simple, then $(S, \Gamma, \leq)$ is regular.

In fact: assume that $(S, \Gamma, \leq)$ is left and right simple. To show that $(S, \Gamma, \leq)$ is regular, let $a \in S$. By (1), $S = (S\Gamma a]$ and $S = (a\Gamma S]$. Then

$$a \in (a\Gamma S] = (a\Gamma(S\Gamma a]] = (a\Gamma S\Gamma a].$$

(3) If $(S, \Gamma, \leq)$ is regular, then the bi-ideals and the subidempotent bi-ideals of $(S, \Gamma, \leq)$ are the same.

To see this, assume that $(S, \Gamma, \leq)$ is regular. Let A be a bi-ideal of $(S, \Gamma, \leq)$. Since $A\Gamma S\Gamma A \subseteq A$, $(A\Gamma S\Gamma A] \subseteq (A] = A$. By assumption, $A \subseteq (A\Gamma S\Gamma A]$, so $A = (A\Gamma S\Gamma A]$. Thus,

$$A\Gamma A = (A\Gamma S\Gamma A]\Gamma(A\Gamma S\Gamma A] \subseteq (A\Gamma S\Gamma A] = A.$$

Theorem. Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. $(S, \Gamma, \leq)$ is left and right simple if and only if $(S, \Gamma, \leq)$ does not contain proper bi-ideals.

Proof. Assume that $(S, \Gamma, \leq)$ is left and right simple. Let A be a bi-ideal of $(S, \Gamma, \leq)$. We shall show that $S \subseteq A$. Let $x \in S$ and $y \in A$. Since S is left simple, we have $S = (y \cup S\Gamma y]$, hence $x \in (y \cup S\Gamma y]$. Then $x \leq y$ or $x \leq z\alpha y$ for some $z \in S$ and for some $\alpha \in \Gamma$. If $x \leq y$, we obtain $x \in A$ by $y \in A$. Assume that $x \leq z\alpha y$. Since S is right simple, $S = (y \cup y\Gamma S]$. Since

$z \in S$, $z \leq y$ or $z \leq y\beta w$ for some $w \in S$ and for some $\beta \in \Gamma$. If $z \leq y$, then $x \leq z\alpha y \leq y\alpha y$. Since $y\alpha y \in A$, $x \in A$. Assume that $z \leq y\beta w$. We have $x \leq z\alpha y \leq y\beta w\alpha y$. Since $y\beta w\alpha y \in A\Gamma S\Gamma A \subseteq A$, it follows that $x \in A$. Thus $S \subseteq A$, therefore, $S = A$.

Conversely, assume that $(S, \Gamma, \leq)$ does not contain proper bi-ideals. To show that $(S, \Gamma, \leq)$ is left simple, let A be a left ideal of $(S, \Gamma, \leq)$. Then A is a bi-ideals of $(S, \Gamma, \leq)$. By assumption, $S = A$. Similarly, if A is a right ideal of $(S, \Gamma, \leq)$, then A is a bi-ideal of $(S, \Gamma, \leq)$, so $S = A$.

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