International Mathematical Forum, Vol. 7, 2012, no. 55, 2719 - 2725

Formulas for the Approximation of the Complete Elliptic Integrals

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Abstract

In this article we give evaluations of the two complete elliptic integrals K and E in the form of Ramanujan's type-$1/\pi$ formulas. The result is a formula for $\Gamma(1/4)^2\pi^{-3/2}$ with accuracy about 120 digits per term.

Keywords: elliptic functions; singular modulus; Ramanujan; Legendre functions; evaluations; constants

1 Introduction

It is known that (see [1],[3]):

$$K(x)=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-x^2\sin^2(\theta)}}=\frac{\pi}{2}{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x^2\right) \tag{1}$$

is the complete elliptic integral of the first kind. The elliptic singular modulus k_r is defined from the equation

$$\frac{K\left(\sqrt{1-k_r^2}\right)}{K(k_r)}=\sqrt{r}. \tag{2}$$

It is known that if $r\in\mathbf{Q}_+^*$, the k_r is algebraic number.
The complete elliptic integral of the second kind is

$$E(x)=\frac{\pi}{2}{}_2F_1\left(\frac{-1}{2},\frac{1}{2};1;x^2\right) \tag{3}$$

and related with $K(x)$ from the formula

$$E(k_r)=\frac{K(k_r)}{\sqrt{r}}\left(\frac{\pi}{3K(k_r)^2}-a(r)\right)+K(k_r). \tag{4}$$

The function $a(r)$ is called elliptic alpha function (see [4]). For $r \in \mathbf{N}$ we set $K[r] = K(k_r)$.
It is known that $K[r]$ can be expressed in terms of products of Γ functions, algebraic numbers and powers of π ([7],[9],[10]). The best way one can obtain that is by using the function

$$b(p) = \frac{\Gamma^2(p)}{\Gamma(2p)}\sqrt{\tan(p\pi)}. \tag{5}$$

It is also known that if $N = n^2 r$, where n and r are positive integers then

$$K[n^2 r] = M_n(r)K[r], \tag{6}$$

where $M_n(r)$ is algebraic.
The following values for $M_n(r)$ are known:

$$M_2(r) = \frac{1+k_r'}{2} \tag{7}$$

$$27M_3^4(r) - 18M_3^2(r) - 8(1-2k_r^2)M_3(r) - 1 = 0 \tag{8}$$

$$(5M_5(r)-1)^5(1-M_5(r)) = 256k_r^2(1-k_r^2)M_5(r) \tag{9}$$

These formulas are for finding $K[4r]$, $K[9r]$, $K[25r]$, which the evaluation of them depend only on knowing k_r and $K[r]$. Note also that only (7) and (8) can be used. The reason is that modular equations of higher degree are not solvable in radicals.
In the present paper we give evaluation formulas of K and E in infinite series using only the elliptic singular modulus k_r at points $q = e^{-\pi\sqrt{r}}$, where r positive real.
Also we give evaluation of the constant

$$\frac{1}{\pi} b\left(\frac{1}{4}\right) = \frac{\Gamma\left(\frac{1}{4}\right)^2}{\pi^{3/2}} \tag{10}$$

in about 120 digits per term formula.
Our methods consists Legendre functions, and we not use the elliptic alpha function $a(r)$.

For a same type series that converging to $1/\pi$ one can see [11].

2 Preliminary Notes

The Legendre P function is defined by

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\nu)}\left(\frac{z+1}{1-z}\right)^{\nu/2} {}_2F_1\left(-\mu, \mu+1; 1-\nu; \frac{1-z}{2}\right) \tag{11}$$

Set

$$\phi(z) = {}_2F_1\left(-\mu, \mu+1; 1-\nu; z\right) = \left(\frac{z}{1-z}\right)^{\nu/2} \Gamma(1-\nu) P_\nu^\mu(1-2z)$$

Then derivating ϕ we have

$$\phi'(z) = \frac{1}{2(1-z)z}\left(\frac{z}{1-z}\right)^{\nu/2}\Gamma(1-\nu)\times$$

$$\times\left[\left(-1-\mu+\nu+2(1+\mu)z\right) P_\nu^\mu(1-2z) + (1+\mu-\nu)P_\nu^{1+\mu}(1-2z)\right] \quad (12)$$

If we assume that

$$\sum_{n=0}^{\infty} \frac{(-\mu)_n\,(1+\mu)_n}{(1-\nu)_n}\frac{z^n}{n!}(\alpha n+\beta) = g \quad (13)$$

then

$$\beta\phi(z) + \alpha z\phi'(z) = g$$

From (11),(12) and (13) we have

Lemma 2.1 *If*

$$\alpha = \frac{2(-1+z)}{-1-\mu+\nu+2z+2\mu z} \quad (14)$$

then

$$\sum_{n=0}^{\infty} \frac{(-\mu)_n\,(1+\mu)_n}{(1-\nu)_n\,n!} z^n(\alpha n+1) = \frac{(-1-\mu+\nu)\left(\frac{z}{1-z}\right)^{\nu/2}\Gamma(1-\nu)P_\nu^{1+\mu}(1-2z)}{-1-\mu+\nu+2(\mu+1)z} \quad (15)$$

Note.
It is also known that if for a given $r \in \mathbf{N}$ the number of fundamental discriminants is $h(-r) = 1$, then (see [9]):

$$K(k_r) = K_r = 2^{1/6}(k_r k_r')^{-24}\sqrt{\frac{2\pi G_r}{r}}, \quad (16)$$

where G_r is a product of Gamma functions.
We know that (see [2], duplication formula):

$$k_{4r} = \frac{1-k_r'}{1+k_r'} \quad (17)$$

Hence in view of (6) and (7)

$$K[16r] = \frac{1+k_{4r}'}{2}K[4r] = \frac{1+k_{4r}'}{2}\frac{1+k_r'}{2}K[r]$$

But

$$k'_{4r} = \sqrt{1-k_{4r}^2} = \sqrt{1-\left(\frac{1-k'_r}{1+k'_r}\right)^2} = \frac{\sqrt{(1+k'_r)^2-(1-k'_r)^2}}{1+k'_r} =$$

or

$$k'_{4r} = \frac{2\sqrt{k'_r}}{1+k'_r} \tag{18}$$

Hence

$$K[16r] = \frac{1+k'_r+2\sqrt{k'_r}}{4}K[r]$$

or

$$K[16r] = \left(\frac{1+\sqrt{k'_r}}{2}\right)^2 K[r] \tag{19}$$

Setting $r \to 4r$ we get

$$K[64r] = \left(\frac{1+\sqrt{k'_{4r}}}{2}\right)^2 K[4r]$$

or equivalently the following useful

Lemma 2.2 *If $r > 0$, then*

$$K[64r] = \frac{\left(\sqrt{1+k'_r}+\sqrt{2\sqrt{k'_r}}\right)^2}{8}K[r] \tag{20}$$

3 Main Results

Theorem 3.1

$$\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n\left(\frac{-1}{2}\right)_n}{(n!)^2}(k_r)^{2n}\left[-4(1-k_r^2)n+1-2k_r^2\right] = \frac{2K(k_r)}{\pi} = 2\vartheta_3^2(q) \tag{21}$$

where $k'_r = \sqrt{1-k_r^2}$, $q = e^{-\pi\sqrt{r}}$.

Proof.
It is known (see [1]), that

$$P_0^{(-1/2)}(1-2z) = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;z\right) \tag{22}$$

hence if we set $\mu = -3/2$ and $\nu = 0$ in Lemma 2.1 we get the result.

The result of the above Theorem is not trivial since the ϑ_3-function can be evaluated from the identity

$$\vartheta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2} \tag{23}$$

in which for this case $q = e^{-\pi\sqrt{r}}$ and the two constants e and π involved.

Theorem 3.2

$$\frac{4E(k_r)}{\pi} = \frac{2K(k_r)}{\pi} + \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2}(k_r)^{2n}[4(1-k_r^2)n + 1 - 2k_r^2] \tag{24}$$

Proof.
The evaluation of $E(k_r)/\pi$ follows if we use the formula:

$$P_{1/2}(1-2z) = \frac{2}{\pi}[2E(z) - K(z)]. \tag{25}$$

Then one can arrive with the same method as in Lemma 2.1 to the desired result.

4 The Application Formula

Set now

$$p = 2 + 216 \cdot 5^{1/4} - 96 \cdot 5^{3/4} \tag{26}$$

then

$$k_{100} = \frac{2-\sqrt{p}}{2+\sqrt{p}} \text{ and } k'_{100} = \frac{2\sqrt{2}p^{1/4}}{2+\sqrt{p}} \tag{27}$$

From the duplication formula is

$$k_{400} = \left(\frac{\sqrt{2} - p^{1/4}}{\sqrt{2} + p^{1/4}}\right)^2 \text{ and } k'_{400} = \frac{2^{7/3}p^{1/8}\sqrt{2+p^{1/2}}}{(\sqrt{2}+p^{1/4})^2}$$

$$k_{1600} = \frac{\left(\sqrt{2}+p^{1/4}\right)^2 - 2\cdot 2^{3/4}p^{1/8}\sqrt{2+\sqrt{p}}}{\left(\sqrt{2}+p^{1/4}\right)^2 + 2\cdot 2^{3/4}p^{1/8}\sqrt{2+\sqrt{p}}}$$

$k_{6400} = w_0 =$

$$= \frac{2 - 2\cdot 2^{5/8}\left(2+\sqrt{p}\right)^{1/4}\sqrt{2\sqrt{2} + 4p^{1/4} + \sqrt{2}\sqrt{p}}p^{1/16} + 2\cdot 2^{3/4}\sqrt{2+\sqrt{p}}p^{1/8} + 2\sqrt{2}p^{1/4} + \sqrt{p}}{2 + 2\cdot 2^{5/8}\left(2+\sqrt{p}\right)^{1/4}\sqrt{2\cdot\sqrt{2} + 4p^{1/4} + \sqrt{2}\sqrt{p}}p^{1/16} + 2\cdot 2^{3/4}\sqrt{2+\sqrt{p}}p^{1/8} + 2\sqrt{2}p^{1/4} + \sqrt{p}}$$

Also from Lemma 2.2 we have

$$K[6400]=\frac{1}{8}\left(\sqrt{1+\frac{2\sqrt{2}p^{1/4}}{2+\sqrt{p}}}+2^{7/8}\left(\frac{p^{1/4}}{2+\sqrt{p}}\right)^{1/4}\right)^{2}K[100]$$

But it is known that

$$K[100]=\frac{4+2\sqrt{5}+\sqrt{2}(3+2\cdot 5^{1/4})}{80}b\left(\frac{1}{4}\right)$$

hence we get the next about 120 digits per term formula for $\frac{1}{\pi}b(1/4)$:

Theorem 4.1

$$\frac{1}{8}\left[4+2\sqrt{5}+\sqrt{2}(3+2\cdot 5^{1/4})\right]^{-1}\left[\sqrt{1+\frac{2\sqrt{2}p^{1/4}}{2+\sqrt{p}}}+2^{7/8}\left(\frac{p^{1/4}}{2+\sqrt{p}}\right)^{1/4}\right]^{-2}\times$$

$$\times\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n\left(\frac{-1}{2}\right)_n}{(n!)^2}(w_0)^{2n}\left[-2(1-w_0^2)n-w_0^2+1/2\right]=\frac{\Gamma\left(\frac{1}{4}\right)}{\pi^{3/2}} \qquad :(a)$$

ACKNOWLEDGEMENTS. I would like to thank Professor M.L. Glasser for his very useful material and the precious time spend for me.

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Received: June, 2012