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Constructions of Some Algebraic Structures from Each Other

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Abstract

We show that various classes of algebraic structures such as paramedial groupoid and AG-groupoid, AG-groupoid and commutative medial, paramedial groupoid and medial, paramedial AG-groupoid and commutative semigroup, AG**-groupoid and commutative semigroup are obtainable from each other.

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1 Introduction

Construction of algebraic structure is always important as well as a challenging task. In this paper we show how and on what conditions some specific groupoids can be obtained from the other known groupoids. Thus we will show paramedial groupoid and AG-groupoid, AG-groupoid and commutative medial, paramedial groupoid and medial, paramedial AG-groupoid and commutative semigroup, AG**-groupoid and commutative semigroup can be obtained from each other either by defining a new operation or through some

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specific endomorphism.
A **groupoid** is a pair $(S, \cdot)$ consisting of a non-empty set S called carrier and a binary operation $(\cdot)$ in S. In other words, $(S, \cdot)$ is a groupoid if $\forall a, b \in S, a \cdot b \in S$. $a \cdot b$ will be simply written as ab. Also we will denote a groupoid by S. An element e of S is called its **left identity** if $ea = a, \forall a \in S$. S is called **commutative** if $ab = ba$, $\forall a, b \in S$. S is called a **semigroup**, if $(\cdot)$ is associative: $a(bc) = (ab)c, \forall a, b, c \in S$. S is called a **Abel Grassmann groupoid**, abbreviated as AG-groupoid, if it satisfies the left invertive law: $(ab)c = (cb)a, \forall a, b, c \in S$. S is called a **medial** if $(\cdot)$ satisfies the medial law: $(ab)(cd) = (ac)(bd), \forall a, b, c, d \in S$. S is called **paramedial** if $(\cdot)$ satisfies the paramedial law: $(ab)(cd) = (db)(ca), \forall a, b, c \in S$. An AG-groupoid S is called **AG*-groupoid** if $(ab)c = b(ac)$, $\forall a, b, c \in S$. An AG-groupoid S is called **AG**-groupoid** if $a(bc) = b(ac)$, $\forall a, b, c \in S$. An AG-groupoid S which is also paramedial is called paramedial AG-groupoid. An AG-groupoid with left identity is called AG-monoid. For some recent study of AG-groupoids we refer to [5, 4].

An AG-groupoid always satisfies the medial law [1, Proposition 2.1].
If S is an **AG*-groupoid** then the following are equivalent:

$$(ab)c = b(ac), (ab)c = b(ca)$$

S with left identity e always satisfies the paramedial law [1].
If S is an AG**-groupoid then $(ab)(cd) = (db)(ca)$ [3].
Every AG-monoid is AG**-groupoid.

2 Construction of algebraic structures from each other

Theorem 2.1. *Let $(S, \cdot)$ be a paramedial groupoid. Define $*$ on S as $x * y = x(cy)$, for some fixed $c \in S$. Then $(S, *)$ is an AG groupoid. In addition if $(S, \cdot)$ satisfies the identity $a(bc) = b(ac)$ then $(S, *)$ becomes AG**-groupoid.*

Proof. Let $x, y, z \in S$. Then by definition of $(*)$, we have

$$\begin{aligned}(x * y) * z &= (x(cy))(cz) \\ &= (z(cy))(cx) \text{ by paramedial law} \\ &= (z * y) * x\end{aligned}$$

Thus $(S, *)$ satisfies left invertive law and hence is an AG-groupoid. Now let $(S, \cdot)$ satisfies $a(bc) = b(ac), \forall a, b, c \in S$. Then by repeated use of $a(bc) = b(ac)$ we have

$$
\begin{aligned}
x * (y * z) &= x(c(y(cz))) \\
&= y(c(x(cz))) \\
&= y * (x * z)
\end{aligned}
$$

Hence $(S, *)$ is an AG**-groupoid. □

Theorem 2.2. *Let $(S, *)$ be an AG-groupoid. Define $(\cdot)$ as $x \cdot y = \alpha(x) * \beta(y)$, where $\alpha, \beta \in End(S)$. Then*

1. *$(S, \cdot)$ is paramedial groupoid if $\alpha^2 = \beta^2$ and if any of the following holds.*
   - *(a) $(S, *)$ is an AG**-groupoid,*
   - *(b) $(S, *)$ is an AG*-groupoid.*
2. *$(S, \cdot)$ is paramedial groupoid if α^2, β^2 are constant.*

Proof. 1. Let $(S, *)$ be an AG-groupoid and

(a) Let $(S, *)$ be an AG**-groupoid and let $a, b, c, d \in S$. Then by definition of $(\cdot)$, we have

$$
\begin{aligned}
(ab)(cd) &= \alpha(\alpha(a) * \beta(b)) * \beta(\alpha(c) * \beta(d)) \\
&= (\alpha^2(a) * \alpha\beta(b)) * (\beta\alpha(c) * \beta^2(d)) \qquad (1)
\end{aligned}
$$

Similarly again by definition of $(\cdot)$, AG*-groupoid and left invertive law, we have

$$
\begin{aligned}
(db)(ca) &= \alpha(\alpha(d) * \beta(b)) * \beta(\alpha(c) * \beta(a)) \\
&= (\alpha^2(d) * \alpha\beta(b)) * (\beta\alpha(c) * \beta^2(a)) \\
&= (\beta\alpha)(c) * [(\alpha^2(d) * \alpha\beta(b)) * \beta^2(a)] \\
&= (\beta\alpha(c) * ((\beta^2(a) * \alpha\beta(b)) * \alpha^2(d)) \\
&= (\beta^2(a) * \alpha\beta(b)) * (\beta\alpha(c) * \alpha^2(d)) \qquad (2)
\end{aligned}
$$

From (1) and (2) $(S, \cdot)$ is paramedial $\iff \alpha^2 = \beta^2$

(b) Let $(S, *)$ be an AG*-groupoid and let $a, b, c, d \in S$. Then by definition of $(\cdot)$, medial law, left invertive law and definition of AG*-

groupoid we have

$$\begin{aligned}(ab)(cd) &= \alpha(\alpha(a)*\beta(b))*\beta(\alpha(c)*\beta(d))\\ &= (\alpha^2(a)*\alpha\beta(b))*(\beta\alpha(c)*\beta^2(d))\\ &= (\alpha^2(a)*\beta\alpha(c))*(\alpha\beta(c)*\beta^2(d))\\ &= [(\alpha\beta(b)*\beta^2(d))*\beta\alpha(c)]*\alpha^2(a))\\ &= [\beta^2(d)*(\alpha\beta(b))*\beta\alpha(c))]*\alpha^2(a))\\ &= [\alpha^2(a)*(\alpha\beta(b))*\beta\alpha(c))]*\beta^2(d))\\ &= (\alpha\beta(b)*\beta\alpha(c))*(\alpha^2(a)*\beta^2(d)) \qquad (3)\end{aligned}$$

Similarly applying left invertive law, medial law and definition of AG*-groupoid, we have

$$\begin{aligned}(db)(ca) &= \alpha(\alpha(d)*\beta(b))*\beta(\alpha(c)*\beta(a))\\ &= (\alpha^2(d)*\alpha\beta(b))*(\beta\alpha(c)*\beta^2(a))\\ &= [(\beta\alpha(c)*\beta^2(a))*\alpha\beta(b)]*\alpha^2(d))\\ &= [(\alpha\beta(b)*\beta^2(a))*\beta\alpha(c)]*\alpha^2(d))\\ &= [\beta^2(a)*(\alpha\beta(b)*\beta\alpha(c))]*\alpha^2(d))\\ &= (\alpha\beta(b)*\beta\alpha(c))*(\beta^2(a)*\alpha^2(d)) \qquad (4)\end{aligned}$$

From (3) and (4), $(S,\cdot)$ is paramedial $\iff \alpha^2=\beta^2$

2. Let $(S,*)$ be an AG-groupoid and let $a,b,c,d\in S$. Then by definition of $(\cdot)$ and left invertive law, we have

$$\begin{aligned}(ab)(cd) &= \alpha(\alpha(a)*\beta(b))*\beta(\alpha(c)*\beta(d))\\ &= (\alpha^2(a)*\alpha\beta(b))*(\beta\alpha(c)*\beta^2(d))\\ &= [(\beta\alpha(c)*\beta^2(d))*\alpha\beta(b)]*\alpha^2(a) \qquad (5)\end{aligned}$$

Similarly by definition of $(\cdot)$ and left invertive law, we have

$$\begin{aligned}(db)(ca) &= \alpha(\alpha(d)*\beta(b))*\beta(\alpha(c)*\beta(a))\\ &= (\alpha^2(a)*\alpha\beta(b))*(\beta\alpha(c)*\beta^2(d))\\ &= [(\beta\alpha(c)*\beta^2(a))*\alpha\beta(b)]*\alpha^2(d) \qquad (6)\end{aligned}$$

From (5) and (6), $(S,\cdot)$ is paramedial if $\alpha^2(a)=\alpha^2(d)$ and $\beta^2(d)=\beta^2(a)$ i.e. α^2 and β^2 are constant.

□

Theorem 2.3. *Let $(S,\cdot)$ be an AG-groupoid. Define $(*)$ as $x*y=(xp)y$ where p is any constant. Then $(S,*)$ is commutative medial.*

Proof. Let $(S,\cdot)$ be an AG-groupoid. and let $a,b,c,d \in S$. Then by definition of $(*)$, we have

$$x * y = (xp)y = (yp)x = y * x$$

Now by repeated use of medial law, we have

$$\begin{aligned}(a*b)*(c*d) &= (((ap)b)p)((cp)d) \\ &= [((ap)b)(cp)](pd) \\ &= [((ap)c)(bp)](pd) \\ &= [((ap)c)p]((bp)d) \\ &= (a*c)*(b*d)\end{aligned}$$

Thus $(S,*)$ is commutative medial. □

Theorem 2.4. *Let $(S,+)$ be commutative medial. Define $(\cdot)$ as $x\cdot y = \alpha(x)+\beta(y)$, $\alpha,\beta \in End(S)$, such that α^2,β are constant. Then $(S,\cdot)$ is AG-groupoid.*

Proof. Let $(S,+)$ be a commutative medial and let $x,y,z \in S$. Then by definition of $(\cdot)$, we have

$$\begin{aligned}(xy)z &= \alpha(\alpha(x)+\beta(y))+\beta(z) \\ &= (\alpha^2(x)+\alpha\beta(y))+\beta(z) \qquad (7)\end{aligned}$$

Similarly,

$$\begin{aligned}(zy)x &= \alpha(\alpha(z)+\beta(y))+\beta(x) \\ &= (\alpha^2(z)+\alpha\beta(y))+\beta(x) \qquad (8)\end{aligned}$$

From (7) and (8), $(S,\cdot)$ is AG-groupoid if and only if $\alpha^2(x)=\alpha^2(z), \beta(x) = \beta(z)$
that is if and only if α^2 and β are constant. □

Theorem 2.5. *Let $(S,\cdot)$ be a paramedial groupoid. Define $(*)$ as $a*b = a(pb)$, where p is any fixed. Then $(S,*)$ is medial.*

Proof. Let $(S,\cdot)$ be a paramedial groupoid and let $a,b,c,d \in S$. Then by definition of $(*)$ and repeated use of paramedial law, we have

$$\begin{aligned}(a*b)*(c*d) &= (a(pb))(p(c(pd))) \\ &= (a((pc))(p(b(pd))) \\ &= (a*c)*(b*d)\end{aligned}$$

Hence $(S,*)$ is medial. □

Theorem 2.6. *Let $(S,*)$ is medial. Define $(\cdot)$ as $x \cdot y = \alpha(x) * \beta(y)$, where $\alpha, \beta \in End(S)$, such that α^2, β^2 are constant. Then $(S,\cdot)$ is paramedial.*

Proof. Let $(S,*)$ be a medial and let $a,b,c,d \in S$. Then by definition of $(\cdot)$, we have

$$\begin{aligned}(ab)\cdot(cd) &= \alpha[\alpha(a) * \beta(b)] * \beta[\alpha(c) * \beta(d)] \\ &= (\alpha^2(a) * \alpha\beta(b)) * (\beta\alpha(c) * \beta^2(d)) \qquad (9)\end{aligned}$$

Similarly

$$\begin{aligned}(db)\cdot(ca) &= \alpha[\alpha(d) * \beta(b)] * \beta[\alpha(c) * \beta(a)] \\ &= (\alpha^2(d) * \alpha\beta(b)) * (\beta\alpha(c) * \beta^2(a)) \qquad (10)\end{aligned}$$

From (9) and (10), $(S,\cdot)$ is paramedial $\iff$ $\alpha^2(a) = \alpha^2(d)$ and $\beta^2(a) = \beta^2(d)$, that is $\iff$ α^2 and β^2 are constant. □

Theorem 2.7. *Let $(S,\cdot)$ be paramedial AG-groupoid. Define $(*)$ as $a * b = (ap)b$, where p is fixed. Then $(S,*)$ is commutative semigroup.*

Proof. Let $a,b,c \in S$. Then by definition of $(*)$ and by left invertive law, we have

$$(a * b) = (ap))b = (bp)a = b * a$$

Now by repeated use of left invertive law and paramedial law

$$\begin{aligned}(a * b) * c &= (((ap)b)p)c \\ &= (cp)((bp)a) \\ &= (ap)((bp)c) \\ &= a * (b * c)\end{aligned}$$

Hence $(S,*)$ is commutative semigroup. □

Theorem 2.8. *Let $(S,+)$ be commutative semigroup. Define $(\cdot)$ as $x \cdot y = \alpha(x) + \beta(y)$, where $\alpha, \beta \in End(S)$, such that $\alpha^2 = \beta^2 = \beta$ or β is constant. Then $(S,\cdot)$ is paramedial AG-groupoid.*

Proof. Let $(S,+)$ be a commutative semigroup and let $a,b,c,\in S$. Then by definition of $(\cdot)$, we have

$$\begin{aligned}(ab)c &= \alpha[\alpha(a) + \beta(b)] + \beta(c) \\ &= (\alpha^2(a) + \alpha\beta(b)) + \beta(c) \qquad (11)\end{aligned}$$

Similarly,

$$\begin{aligned}(cb)a &= \alpha[\alpha(c) + \beta(b)] + \beta(a) \\ &= (\alpha^2(c) + \alpha\beta(b)) + \beta(a) \qquad (12)\end{aligned}$$

From (11) and (12),

$$(ab)c = (cb)a \iff \alpha^2(a) + \beta(c) \ = \ \alpha^2(c) + \beta(a)$$

$\iff$ (1) $\alpha^2(a) = \alpha^2(c)$ and $\beta(a) = \beta(c)$
$\iff$ α^2 and β are constant
and

$$\begin{aligned}(ab)(cd) &= \alpha[\alpha(a) + \beta(b)] + \beta[\alpha(c) + \beta(d)] \\ &= \alpha^2(a) + \alpha\beta(b) + \beta\alpha(c) + \beta^2(d)\end{aligned}$$

Similarly,

$$\begin{aligned}(db) \cdot (ca) &= \alpha[\alpha(d) + \beta(b)] + \beta[\alpha(c) + \beta(a)] \\ &= \alpha^2(d) + \alpha\beta(b)] + \beta\alpha(c) + \beta^2(a)] \\ (ab) \cdot (cd) &= (db) \cdot (ca) \iff \alpha^2(a) + \beta^2(d) = \alpha^2(d) + \beta^2(a)\end{aligned}$$

$$\iff (3)\ \alpha^2 = \beta^2 \text{ or } (4)\ \alpha^2 \text{ and } \beta^2 \text{ are constant.}$$

Thus the claim holds $\iff$ $\beta^2 = \beta$ or β is constant. □

Theorem 2.9. *Let $(S, \cdot)$ be an AG^{**}-groupoid. Define $(*)$ as $x * y = (xc)y$ where c is fixed. Then $(S, *)$ is commutative semigroup.*

Proof. Let $(S, \cdot)$ be an AG**-groupoid and let $x, y, z \in S$. Then by definition of $(*)$, we have

$$x * y = (xp)y = (yp)x = y * x$$

Now by left invertive law and definition of AG**-groupoid, we have

$$\begin{aligned}(x * y) * z &= (((xc)y)c)z \\ &= (zc)((xc)y) \\ &= (xc)((zc)y) \\ &= (xc)((yc)z) \\ &= x * (y * z))\end{aligned}$$

Thus $(S, *)$ is commutative semigroup. □

Theorem 2.10. *Let $(S, +)$ be commutative semigroup. Define $(\cdot)$ as $a \cdot b = \alpha(a) + \beta(b)$, where $\alpha, \beta \in End(S)$, such that $\alpha^2 = I$ or β and β is constant. Then $(S, \cdot)$ is AG^{**}-groupoid.*

Proof. Let $(S,+)$ is commutative semigroup and let $a,b,c,\in S$. Then by definition of $(\cdot)$, we have

$$\begin{aligned}(ab)\cdot c &= \alpha[\alpha(a)+\beta(b)]+\beta(c)] \\ &= (\alpha^2(a)+\alpha\beta(b))+\beta(c)) \qquad (13)\end{aligned}$$

Similarly by commutativity and associativity property of $(S,+)$

$$\begin{aligned}(cb)\cdot a &= \alpha[\alpha(c)+\beta(b)]+\beta(a) \\ &= (\alpha^2(c)+\alpha\beta(b))+\beta(a) \qquad (14) \\ &= (\beta(a)+\alpha\beta(b))+\alpha^2(c) \qquad (15)\end{aligned}$$

From (13) and (15)

$$(S,\cdot) \text{ is an AG-groupoid } \iff \alpha^2=\beta \text{ or } \alpha^2 \text{ and } \beta \text{ are constant.}$$

Next

$$\begin{aligned}a(bc) &= \alpha(a)+\beta(\alpha(b)+\beta(c)) \\ &= \alpha(a)+(\beta\alpha(b)+\beta^2(c)) \qquad (16)\end{aligned}$$

Similarly by commutativity and associativity of $(S,+)$

$$\begin{aligned}b(ac) &= \alpha(b)+\beta(\alpha(a)+\beta(c) \\ &= \alpha(b)+(\beta\alpha(a)+\beta^2(c)) \\ &= \beta\alpha(a)+(\alpha(b)+\beta^2(c) \qquad (17)\end{aligned}$$

From(16) and (17) $a(bc)=b(ac) \iff \beta=\alpha$ or α and $\beta\alpha$ are constant.

Hence $(S,\cdot)$ is an AG**-groupoid. □

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