

Further Results Related to Cauchy's Proper Bound for the Zeros of Entire Functions

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Abstract

A single valued function of one complex variable which is analytic in the finite complex plane is called an entire function. In this paper we would like to establish the bounds for the moduli of zeros of entire functions. Some examples are provided to clear the notions.

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1 Introduction, Definitions and Notations.

Let

$$P(z) = a_0 + a_1z + a_2z^2 + a_3z^3 + \dots + a_{n-1}z^{n-1} + a_nz^n; |a_n| \neq 0$$

be a polynomial of degree n . Datt and Govil[2]; Govil and Rahaman[4]; Marden[8]; Mohammad[9]; Chattopadhyay, Das, Jain and Konwer[1]; Joyal, Labelle and Rahaman[5]; Jain{[6],[7]}; Sun and Hsieh[10]; Zilovic, Roytman, Combettes and Swamy[12]; Das and Datta[3] etc. worked in the theory of the distribution of the zeros of polynomials and obtained some newly developed results.

In this paper we intend to establish some sharper results concerning the theory of distribution of zeros of entire functions.

2 Lemma.

In this section we present a lemma which will be needed in the sequel.

Lemma 1 [11] *If $f(z)$ is an entire function of order ρ then for every $\epsilon > 0$ and for all sufficiently large r the inequality $N(r) \leq r^{\rho+\epsilon}$ holds where $N(r)$ is the number of zeros of $f(z)$ in $|z| \leq r$.*

3 Theorems.

Theorem 1 *Let $P(z)$ be an entire function defined by*

$$P(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n + \dots$$

whose order ρ is finite. Also for all sufficiently large r in the disc $|z| \leq r$, $|a_{N(r)}| \neq 0$, $|a_0| \neq 0$. and also $a_n \rightarrow 0$ as $n > N(r)$. Then all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{t'_0} \leq |z| \leq t_0$$

where t_0 is the greatest positive root of

$$g(t) \equiv |a_{N(r)}| t^{N(r)+1} - (|a_{N(r)}| + M) t^{N(r)} + M = 0$$

and t'_0 is the greatest positive root of

$$\begin{aligned} f(t) &\equiv |a_0| t^{N(r)+1} - (|a_0| + M') t^{N(r)} + M' = 0 \\ \text{where } M &= \max \{ |a_0|, |a_1|, \dots, |a_{N(r)-1}| \} \\ \text{and } M' &= \max \{ |a_1|, |a_2|, \dots, |a_{N(r)}| \}. \end{aligned}$$

Proof. Now

$$P(z) \approx a_0 + a_1 z + a_2 z^2 + \dots + a_{N(r)} z^{N(r)}$$

because $N(r)$ exists for $|z| \leq r$; r is sufficiently large and $a_n \rightarrow 0$ as $n > N(r)$. Then all the zeros of $P(z)$ lie in the ring shaped region given in Theorem 1 which we are to prove.

Now

$$\begin{aligned} |P(z)| &\approx |a_0 + a_1 z + a_2 z^2 + \dots + a_{N(r)} z^{N(r)}| \\ &\geq |a_{N(r)}| |z|^{N(r)} - |a_0 + a_1 z + a_2 z^2 + \dots + a_{N(r)-1} z^{N(r)-1}|. \end{aligned}$$

Also

$$\begin{aligned} &|a_0 + a_1 z + a_2 z^2 + \dots + a_{N(r)-1} z^{N(r)-1}| \\ &\leq |a_0| + \dots + |a_{N(r)-1}| |z|^{N(r)-1} \\ &\leq M (1 + |z| + \dots + |z|^{N(r)-1}) \\ &= M \frac{|z|^{N(r)} - 1}{|z| - 1} \text{ if } |z| \neq 1. \end{aligned} \tag{1}$$

Therefore using (1) we obtain that

$$\begin{aligned} |P(z)| &\geq |a_{N(r)}| |z|^{N(r)} - |a_0 + a_1 z + a_2 z^2 + \dots + a_{N(r)-1} z^{N(r)-1}| \\ &\geq |a_{N(r)}| |z|^{N(r)} - M \frac{|z|^{N(r)} - 1}{|z| - 1}. \end{aligned}$$

Hence

$$\begin{aligned}
 |P(z)| \geq 0 \text{ if } |a_{N(r)}| |z|^{N(r)} - M \frac{|z|^{N(r)} - 1}{|z| - 1} &> 0 \\
 \text{i.e., if } |a_{N(r)}| |z|^{N(r)} &> M \frac{|z|^{N(r)} - 1}{|z| - 1} \\
 \text{i.e., if } |a_{N(r)}| |z|^{N(r)+1} - |a_{N(r)}| |z|^{N(r)} &> M (|z|^{N(r)} - 1) \\
 \text{i.e., if } |a_{N(r)}| |z|^{N(r)+1} - |a_{N(r)}| |z|^{N(r)} - M |z|^{N(r)} + M &> 0 \\
 \text{i.e., if } |a_{N(r)}| |z|^{N(r)+1} - (|a_{N(r)}| + M) |z|^{N(r)} + M &> 0.
 \end{aligned}$$

Therefore on $|z| \neq 1$,

$$|P(z)| \geq 0 \text{ if } |a_{N(r)}| |z|^{N(r)+1} - (|a_{N(r)}| + M) |z|^{N(r)} + M > 0.$$

Now let us consider

$$g(t) \equiv |a_{N(r)}| t^{N(r)+1} - (|a_{N(r)}| + M) t^{N(r)} + M = 0. \quad (2)$$

Clearly the maximum number of changes in sign in (2) is two. So the maximum number of positive roots of $g(t) = 0$ is two and by Descartes' rule of sign if it is less, less by two. Clearly $t = 1$ is one positive root of (2). So $g(t) = 0$ must have another positive root t_1 (say).

Let us take $t_0 = \max \{1, t_1\}$. Clearly for $t > t_0$, $g(t) > 0$. If not, for some $t = t_2 > t_0$, $g(t_2) < 0$.

Now $g(t_2) < 0$ and $g(\infty) > 0$ imply that $g(t) = 0$ has another positive root in (t_2, ∞) which gives a contradiction.

Therefore for $t > t_0$, $g(t) > 0$ and so $t_0 > 1$.

Hence $|P(z)| \geq 0$ for $|z| > t_0$.

$$\text{Therefore all the zeros of } P(z) \text{ lie in the disc } |z| \leq t_0. \quad (3)$$

Again let us consider

$$\begin{aligned}
 Q(z) &= z^{N(r)} P\left(\frac{1}{z}\right) \\
 &\approx z^{N(r)} \left\{ a_0 + \frac{a_1}{z} + \dots + \frac{a_{N(r)}}{z^{N(r)}} \right\} \\
 &= a_0 z^{N(r)} + a_1 z^{N(r)-1} + \dots + a_{N(r)} \\
 \text{i.e., } |Q(z)| &\geq |a_0| |z|^{N(r)} - |a_1 z^{N(r)-1} + \dots + a_{N(r)}| \text{ for } |z| \neq 1.
 \end{aligned}$$

Now

$$\begin{aligned}
 |a_1 z^{N(r)-1} + \dots + a_{N(r)}| &\leq |a_1| |z|^{N(r)-1} + \dots + |a_{N(r)}| \\
 &\leq M' (|z|^{N(r)-1} + \dots + 1) \\
 &= M' \left(\frac{|z|^{N(r)} - 1}{|z| - 1} \right) \text{ for } |z| \neq 1. \quad (4)
 \end{aligned}$$

Using (4) we get that

$$\begin{aligned}
 |Q(z)| &\geq |a_0| |z|^{N(r)} - |a_1 z^{N(r)-1} + \dots + a_{N(r)}| \\
 &\geq |a_0| |z|^{N(r)} - M' \left(\frac{|z|^{N(r)} - 1}{|z| - 1} \right) \text{ for } |z| \neq 1.
 \end{aligned}$$

Therefore for $|z| \neq 1$,

$$\begin{aligned}
 |Q(z)| &\geq 0 \text{ if } |a_0| |z|^{N(r)} - M' \left(\frac{|z|^{N(r)} - 1}{|z| - 1} \right) > 0 \\
 \text{i.e., if } &|a_0| |z|^{N(r)} > M' \left(\frac{|z|^{N(r)} - 1}{|z| - 1} \right) \\
 \text{i.e., if } &|a_0| |z|^{N(r)+1} - |a_0| |z|^{N(r)} - M' |z|^{N(r)} + M' > 0 \\
 \text{i.e., if } &|a_0| |z|^{N(r)+1} - (|a_0| + M') |z|^{N(r)} + M' > 0.
 \end{aligned}$$

So for $|z| \neq 1$,

$$|Q(z)| \geq 0 \text{ if } |a_0| |z|^{N(r)+1} - (|a_0| + M') |z|^{N(r)} + M' > 0.$$

Let us consider

$$f(t) \equiv |a_0| t^{N(r)+1} - (|a_0| + M') t^{N(r)} + M' = 0.$$

Since the maximum number of changes of sign in $f(t)$ is two, the maximum number of positive roots of $f(t) = 0$ is two and by Descartes' rule of sign if it is less, less by two. Clearly $t = 1$ is one positive root of $f(t) = 0$. So $f(t) = 0$ must have another positive root .

Let us take $t'_0 = \text{Max}\{1, t_2\}$. Clearly for $t > t'_0$, $f(t) > 0$. If not, for some $t_3 > t'_0$, $f(t_3) < 0$. Now $f(t_3) < 0$ and $f(\infty) > 0$ implies that $f(t) = 0$ have another positive root in the interval (t_3, ∞) which is a contradiction.

Therefore for $t > t'_0$, $f(t) > 0$.

Also $t'_0 \geq 1$. So $|Q(z)| \geq 0$ for $|z| > t'_0$.

Therefore $Q(z)$ does not vanish in $|z| > t'_0$.

Hence all the zeros of $Q(z)$ lie in $|z| \leq t'_0$.

Let $z = z_0$ be a zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$.

Putting $z = \frac{1}{z_0}$ in $Q(z)$ we get that

$$Q\left(\frac{1}{z_0}\right) = \left(\frac{1}{z_0}\right)^{N(r)} P(z_0) = \left(\frac{1}{z_0}\right)^{N(r)} \cdot 0 = 0.$$

Therefore $Q\left(\frac{1}{z_0}\right) = 0$. So $z = \frac{1}{z_0}$ is a root of $Q(z) = 0$. Hence $\left|\frac{1}{z_0}\right| \leq t'_0$ implies that $|z_0| \geq \frac{1}{t'_0}$.

As z_0 is an arbitrary root of $P(z) = 0$.

$$\text{Therefore all the zeros of } P(z) \text{ lie in } |z| \geq \frac{1}{t'_0}. \quad (5)$$

From (3) and (5) we get that all the zeros of $P(z)$ lie in the proper ring shaped region

$$\frac{1}{t'_0} \leq |z| \leq t_0$$

where t_0 and t'_0 are the greatest positive roots of the equations

$$g(t) \equiv |a_{N(r)}| t^{N(r)+1} - (|a_{N(r)}| + M) t^{N(r)} + M = 0$$

and

$$f(t) \equiv |a_0| t^{N(r)+1} - (|a_0| + M') t^{N(r)} + M' = 0$$

where M and M' are given in the statement of Theorem 1. ■

Remark 2 The limit in Theorem 1 is attained by $P(z) = z^2 - z - 1$. Here $\rho = 0$ and $N(r) = 2 \leq r^{0+\epsilon} = r^\epsilon$. For $\epsilon > 0$ and sufficiently large r , all $a_n = 0, n \geq 2$. Also $a_0 = -1, a_1 = -1, a_2 = 1$.

Therefore

$$M = \text{Max} \{|a_0|, |a_1|\} = 1 \text{ and } M' = \text{Max} \{|a_1|, |a_2|\} = 1$$

and

$$\begin{aligned} g(t) &\equiv |a_2| t^3 - (|a_2| + M) t^2 + M = 0 \\ \text{i.e., } g(t) &\equiv t^3 - (1 + 1) t^2 + 1 = 0 \\ \text{i.e., } g(t) &\equiv t^3 - 2t^2 + 1 = 0. \end{aligned}$$

Again

$$\begin{aligned} f(t) &\equiv |a_0| t^3 - (|a_0| + M') t^2 + M' = 0 \\ \text{i.e., } f(t) &\equiv 1.t^3 - (1 + 1) t^2 + 1 = 0 \\ \text{i.e., } f(t) &\equiv t^3 - 2t^2 + 1 = 0. \end{aligned}$$

So $f(t) = 0$ and $g(t) = 0$ represent the same equation. Maximum number of positive roots of $f(t) = 0$ and $g(t) = 0$ are same. Now

$$\begin{aligned} g(t) &= 0 \\ \text{implies that } t^3 - 2t^2 + 1 &= 0 \\ \text{i.e., } (t - 1)(t^2 - t - 1) &= 0. \end{aligned}$$

Therefore

$$t = 1 \text{ and } t = \frac{1 \pm \sqrt{(-1)^2 - 4.1.(-1)}}{2.1} = \frac{1 \pm \sqrt{3}}{2}.$$

Hence the positive roots of $g(t) = 0$ are 1 and $\frac{1+\sqrt{3}}{2}$. So

$$t_0 = \max \left\{ 1, \frac{1 + \sqrt{3}}{2} \right\} = \frac{1 + \sqrt{3}}{2}.$$

Also the maximum positive root of $f(t) = 0$ is

$$t'_0 = \max \left\{ 1, \frac{1 + \sqrt{3}}{2} \right\} = \frac{1 + \sqrt{3}}{2}.$$

So in view of Theorem 1 all the zeros of $P(z)$ lie in

$$\begin{aligned} \frac{1}{t'_0} &\leq |z| \leq t_0 \\ \text{i.e., } \frac{1}{\frac{1+\sqrt{3}}{2}} &\leq |z| \leq \frac{1+\sqrt{3}}{2} \\ \text{i.e., } \frac{\sqrt{3}-1}{2} &\leq |z| \leq \frac{1+\sqrt{3}}{2}. \end{aligned}$$

Now the zeros of $P(z)$ are given by solving $z^2 - z - 1 = 0$. Therefore $z = \frac{1 \pm \sqrt{3}}{2}$. Let us denote the zeros of $P(z)$ by $z_1 = \frac{1 + \sqrt{3}}{2}$ and $z_2 = \frac{1 - \sqrt{3}}{2} = -\frac{\sqrt{3}-1}{2}$. Clearly z_1 lies on the upper boundary and z_2 lies on the lower boundary. So the best possible result is given by $P(z) = z^2 - z - 1$.

Theorem 3 Let $P(z)$ be an entire function defined by

$$P(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n + \dots$$

with finite order ρ , $a_{N(r)} \neq 0$, $a_0 \neq 0$ and also $a_n \rightarrow 0$ for $n > N(r)$ for the disc $|z| \leq r$ when r is sufficiently large. Further for some $\rho > 0$,

$$|a_0| \rho^{N(r)} \geq |a_1| \rho^{N(r)-1} \geq \dots \geq |a_{N(r)-1}| \rho \geq |a_{N(r)}|.$$

Then all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{\rho \left(1 + \frac{|a_1|}{|a_0| \rho}\right)} < |z| < \frac{1}{\rho} \left(1 + \frac{|a_0|}{|a_{N(r)}|} \rho^{N(r)}\right).$$

Proof. For the given entire function

$$P(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n + \dots$$

with $a_n \rightarrow 0$ as $n > N(r)$, where r is sufficiently large, $N(r)$ exists and $N(r) \leq r^{k+\epsilon}$.

Therefore

$$P(z) \approx a_0 + a_1z + a_2z^2 + \dots + a_{N(r)}z^{N(r)}$$

as $a_0 \neq 0$, $a_{N(r)} \neq 0$ and $a_n \rightarrow 0$ for $n > N(r)$.

Let us consider

$$\begin{aligned} R(z) &= \rho^{N(r)} P\left(\frac{z}{\rho}\right) \\ &\approx \rho^{N(r)} \left(a_0 + a_1 \frac{z}{\rho} + a_2 \frac{z^2}{\rho^2} + \dots + a_{N(r)} \frac{z^{N(r)}}{\rho^{N(r)}} \right) \\ &= (a_0 \rho^{N(r)} + a_1 \rho^{N(r)-1} z + \dots + a_{N(r)} z^{N(r)}). \end{aligned}$$

Therefore

$$|R(z)| \geq |a_{N(r)}| |z|^{N(r)} - |a_0 \rho^{N(r)} + a_1 \rho^{N(r)-1} z + \dots + a_{N(r)-1} \rho z^{N(r)-1}|. \quad (6)$$

Now by the given condition $|a_0| \rho^{N(r)} \geq |a_1| \rho^{N(r)-1} \geq \dots$ provided $|z| \neq 0$, we obtain that

$$\begin{aligned} & |a_0 \rho^{N(r)} + a_1 \rho^{N(r)-1} z + \dots + a_{N(r)-1} \rho z^{N(r)-1}| \\ & \leq |a_0| \rho^{N(r)} + \dots + |a_{N(r)-1}| \rho |z|^{N(r)-1} \\ & \leq |a_0| \rho^{N(r)} |z|^{N(r)} \left(\frac{1}{|z|} + \dots + \frac{1}{|z|^{N(r)}} \right). \end{aligned}$$

Therefore on $|z| \neq 0$,

$$\begin{aligned} & -|a_0 \rho^{N(r)} + a_1 \rho^{N(r)-1} z + \dots + a_{N(r)-1} \rho z^{N(r)-1}| \\ & \geq -|a_0| \rho^{N(r)} |z|^{N(r)} \left(\frac{1}{|z|} + \dots + \frac{1}{|z|^{N(r)}} \right). \end{aligned} \quad (7)$$

Therefore using (7) we get from (6) that

$$\begin{aligned} |R(z)| & \geq |a_{N(r)}| |z|^{N(r)} - |a_0| \rho^{N(r)} |z|^{N(r)} \left(\frac{1}{|z|} + \dots + \frac{1}{|z|^{N(r)}} \right) \\ & \geq |a_{N(r)}| |z|^{N(r)} - |a_0| \rho^{N(r)} |z|^{N(r)} \left(\frac{1}{|z|} + \dots + \frac{1}{|z|^{N(r)}} + \dots \right) \\ & = |z|^{N(r)} \left[|a_{N(r)}| - |a_0| \rho^{N(r)} \left\{ \sum_{k=1}^{\infty} \frac{1}{|z|^k} \right\} \right]. \end{aligned}$$

Clearly $\sum_{k=1}^{\infty} \frac{1}{|z|^k}$ is a geometric series which is convergent for $\frac{1}{|z|} < 1$ i.e., for $|z| > 1$ and converges to

$$\frac{1}{|z|} \frac{1}{1 - \frac{1}{|z|}} = \frac{1}{|z| - 1}.$$

Therefore

$$\sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z| - 1} \text{ if } |z| > 1.$$

Hence we get from above that for $|z| > 1$

$$|R(z)| > |z|^{N(r)} \left(|a_{N(r)}| - \rho^{N(r)} |a_0| \frac{1}{|z| - 1} \right).$$

Now for $|z| > 1$,

$$\begin{aligned}
 |R(z)| > 0 \text{ if } |z|^{N(r)} \left(|a_{N(r)}| - \rho^{N(r)} |a_0| \frac{1}{|z| - 1} \right) &\geq 0 \\
 \text{i.e., if } |a_{N(r)}| - \rho^{N(r)} |a_0| \frac{1}{|z| - 1} &\geq 0 \\
 \text{i.e., if } |a_{N(r)}| \geq \rho^{N(r)} \frac{|a_0|}{|z| - 1} \\
 \text{i.e., if } |z| - 1 \geq \rho^{N(r)} \frac{|a_0|}{|a_{N(r)}|} \\
 \text{i.e., if } |z| \geq 1 + \rho^{N(r)} \frac{|a_0|}{|a_{N(r)}|} &> 1.
 \end{aligned}$$

Therefore

$$|R(z)| > 0 \text{ if } |z| \geq 1 + \rho^{N(r)} \frac{|a_0|}{|a_{N(r)}|}.$$

So all the zeros of $R(z)$ lie in

$$|z| < 1 + \frac{|a_0|}{|a_{N(r)}|} \rho^{N(r)}.$$

Let z_0 be an arbitrary zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $z_0 \neq 0$. Putting $z = \rho z_0$ in $R(z)$ we have

$$R(\rho z_0) = \rho^{N(r)} P(z_0) = \rho^{N(r)} . 0 = 0.$$

Hence $z = \rho z_0$ is a zero of $R(z)$. Therefore

$$\begin{aligned}
 |\rho z_0| &< 1 + \frac{|a_0|}{|a_{N(r)}|} \rho^{N(r)} \\
 \text{i.e., } |z_0| &< \frac{1}{\rho} \left(1 + \frac{|a_0|}{|a_{N(r)}|} \rho^{N(r)} \right).
 \end{aligned}$$

Since z_0 is any zero of $P(z)$ therefore all the zeros of $P(z)$ lie in

$$|z| < \frac{1}{\rho} \left(1 + \frac{|a_0|}{|a_{N(r)}|} \rho^{N(r)} \right). \quad (8)$$

Again let us consider

$$F(z) = \rho^{N(r)} z^{N(r)} P\left(\frac{1}{\rho z}\right).$$

Now

$$\begin{aligned} F(z) &= \rho^{N(r)} z^{N(r)} P\left(\frac{1}{\rho z}\right) \\ &\approx \rho^{N(r)} z^{N(r)} \left\{ a_0 + \frac{a_1}{\rho z} + \dots + \frac{a_{N(r)}}{(\rho z)^{N(r)}} \right\} \\ &= a_0 \rho^{N(r)} z^{N(r)} + a_1 \rho^{N(r)-1} z^{N(r)-1} + \dots + a_{N(r)}. \end{aligned}$$

Therefore

$$|F(z)| \geq |a_0| \rho^{N(r)} |z|^{N(r)} - |a_1 \rho^{N(r)-1} z^{N(r)-1} + \dots + a_{N(r)}|.$$

Again

$$\begin{aligned} |a_1 \rho^{N(r)-1} z^{N(r)-1} + \dots + a_{N(r)}| &\leq |a_1| \rho^{N(r)-1} |z|^{N(r)-1} + \dots + |a_{N(r)}| \\ &\leq |a_1| \rho^{N(r)-1} \left(|z|^{N(r)-1} + \dots + |z| + 1 \right) \end{aligned}$$

provided $|z| \neq 0$. So

$$a_1 \rho^{N(r)-1} z^{N(r)-1} + \dots + |a_{N(r)}| \leq |a_1| \rho^{N(r)-1} |z|^{N(r)} \left(\frac{1}{|z|} + \dots + \frac{1}{|z|^{N(r)}} \right).$$

So for $|z| \neq 0$,

$$\begin{aligned} |F(z)| &\geq |a_0| \rho^{N(r)} |z|^{N(r)} - |a_1| \rho^{N(r)-1} |z|^{N(r)} \left(\frac{1}{|z|} + \dots + \frac{1}{|z|^{N(r)}} \right) \\ &= \rho^{N(r)-1} |z|^{N(r)} \left[|a_0| \rho - |a_1| \left(\frac{1}{|z|} + \dots + \frac{1}{|z|^{N(r)}} \right) \right]. \end{aligned}$$

Therefore for $|z| \neq 0$,

$$|F(z)| > \rho^{N(r)-1} |z|^{N(r)} \left[|a_0| \rho - |a_1| \sum_{k=1}^{\infty} \frac{1}{|z|^k} \right]. \quad (9)$$

The geometric series $\sum_{k=1}^{\infty} \frac{1}{|z|^k}$ is convergent for

$$\begin{aligned} \frac{1}{|z|} &< 1 \\ \text{i.e., for } |z| &> 1 \end{aligned}$$

and converges to

$$\frac{1}{|z|} \frac{1}{1 - \frac{1}{|z|}} = \frac{1}{|z| - 1}.$$

Therefore

$$\sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z| - 1} \text{ if } |z| > 1. \quad (10)$$

Using (9) and (10) we have for $|z| > 1$,

$$|F(z)| > \rho^{N(r)-1} |z|^{N(r)} \left[|a_0| \rho - \frac{|a_1|}{|z| - 1} \right].$$

Hence for $|z| > 1$,

$$|F(z)| > 0 \text{ if } |z|^{N(r)} \rho^{N(r)-1} \left[|a_0| \rho - \frac{|a_1|}{|z| - 1} \right] \geq 0$$

$$\text{i.e., if } |a_0| \rho - \frac{|a_1|}{|z| - 1} \geq 0$$

$$\text{i.e., if } |a_0| \rho \geq \frac{|a_1|}{|z| - 1}$$

$$\text{i.e., if } |z| \geq 1 + \frac{|a_1|}{|a_0| \rho} > 1.$$

Therefore

$$|F(z)| > 0 \text{ for } |z| > 1 + \frac{|a_1|}{|a_0| \rho}.$$

So $F(z)$ does not vanish in

$$|z| \geq 1 + \frac{|a_1|}{|a_0| \rho}.$$

Equivalently all the zeros of $F(z)$ lie in

$$|z| < 1 + \frac{|a_1|}{|a_0| \rho}.$$

Let $z = z_0$ be any zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $a_0 \neq 0$ and $z_0 \neq 0$.

Now let us put $z = \frac{1}{\rho z_0}$ in $F(z)$. So we have

$$\begin{aligned} F\left(\frac{1}{\rho z_0}\right) &= \rho^{N(r)} \left(\frac{1}{\rho z_0}\right)^{N(r)} P(z_0) \\ &= \left(\frac{1}{z_0}\right)^{N(r)} \cdot 0 = 0. \end{aligned}$$

Therefore $z = \frac{1}{\rho z_0}$ is a root of $F(z)$.

Hence

$$\begin{aligned} \left| \frac{1}{\rho z_0} \right| &< 1 + \frac{|a_1|}{|a_0|\rho} \\ \text{i.e., } \left| \frac{1}{z_0} \right| &< \rho \left(1 + \frac{|a_1|}{|a_0|\rho} \right) \\ \text{i.e., } |z_0| &> \frac{1}{\rho \left(1 + \frac{|a_1|}{|a_0|\rho} \right)}. \end{aligned}$$

As z_0 is an arbitrary zero of $P(z)$, all the zeros of $P(z)$ lie on

$$|z| > \frac{1}{\rho \left(1 + \frac{|a_1|}{|a_0|\rho} \right)}. \quad (11)$$

From (8) and (11) we get that all the zeros of $P(z)$ lie on the proper ring shaped region

$$\frac{1}{\rho \left(1 + \frac{|a_1|}{|a_0|\rho} \right)} < |z| < \frac{1}{\rho} \left(1 + \frac{|a_0|}{|a_{N(r)}|} \rho^{N(r)} \right)$$

where

$$|a_0| \rho^{N(r)} \geq |a_1| \rho^{N(r)-1} \geq \dots \geq |a_{N(r)}|$$

for some $\rho > 0$. ■

Corollary 4 *From Theorem 2 we can easily conclude that all the zeros of*

$$P(z) = a_0 + a_1 z + \dots + a_n z^n$$

of degree n , $|a_n| \neq 0$ with the property $|a_0| \geq |a_1| \geq \dots \geq |a_n|$ lie in the proper ring shaped region

$$\frac{1}{\left(1 + \frac{|a_1|}{|a_0|} \right)} < |z| < \left(1 + \frac{|a_0|}{|a_n|} \right)$$

just on putting $\rho = 1$.

Theorem 5 *Let $P(z)$ be an entire function with finite order ρ . For sufficiently large values of r in the disk $|z| \leq r$, the Taylor's series expansion of $P(z)$*

$$P(z) = a_0 + a_{p_1} z^{p_1} + a_{p_2} z^{p_2} + \dots + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)}, a_0 \neq 0$$

be such that $1 \leq p_1 < p_2 \dots < p_m \leq N(r) - 1$, p_i 's are integers and for some $\rho > 0$,

$$|a_0| \rho^{N(r)} \geq |a_{p_1}| \rho^{N(r)-p_1} \geq \dots \geq |a_{p_m}| \rho^{N(r)-p_m}.$$

Then all the zeros of $P(z)$ lie in the proper ring shaped region

$$\frac{1}{\rho t_0} < |z| < \frac{1}{\rho} t_0$$

where t_0 and t'_0 are the unique positive roots of the equations

$$g(t) \equiv |a_{N(r)}| t^{N(r)-p_m} - |a_{N(r)}| t^{N(r)-p_m-1} - |a_0| \rho^{N(r)} = 0 \text{ and}$$

$$f(t) \equiv |a_0| \rho^{p_1} t^{p_1} - |a_0| \rho^{p_1} t^{p_1-1} - |a_{p_1}| = 0$$

respectively.

Proof. Let

$$P(z) = a_0 + a_{p_1} z^{p_1} + \dots + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)}, |a_{N(r)}| \neq 0. \quad (12)$$

Also for some $\rho > 0$,

$$|a_0| \rho^{N(r)} \geq |a_{p_1}| \rho^{N(r)-p_1} \geq \dots \geq |a_{p_m}| \rho^{N(r)-p_m}.$$

Let us consider

$$\begin{aligned} R(z) &= \rho^{N(r)} P\left(\frac{z}{\rho}\right) \\ &= \rho^{N(r)} \left\{ a_0 + a_{p_1} \frac{z^{p_1}}{\rho^{p_1}} + \dots + a_{p_m} \frac{z^{p_m}}{\rho^{p_m}} + a_{N(r)} \frac{z^{N(r)}}{\rho^{N(r)}} \right\} \\ &= a_0 \rho^{N(r)} + a_{p_1} \rho^{N(r)-p_1} z^{p_1} + \dots + a_{p_m} \rho^{N(r)-p_m} z^{p_m} + a_{N(r)} z^{N(r)}. \end{aligned}$$

Therefore

$$\begin{aligned} |R(z)| &\geq |a_{N(r)} z^{N(r)}| \\ &\quad - |a_0 \rho^{N(r)} + a_{p_1} \rho^{N(r)-p_1} z^{p_1} + \dots + a_{p_m} \rho^{N(r)-p_m} z^{p_m}|. \end{aligned} \quad (13)$$

Now for $|z| \neq 0$,

$$\begin{aligned} &|a_0 \rho^{N(r)} + a_{p_1} \rho^{N(r)-p_1} z^{p_1} + \dots + a_{p_m} \rho^{N(r)-p_m} z^{p_m}| \\ &\leq |a_0| \rho^{N(r)} + |a_{p_1}| \rho^{N(r)-p_1} |z|^{p_1} + \dots + |a_{p_m}| \rho^{N(r)-p_m} |z|^{p_m} \\ &\leq |a_0| \rho^{N(r)} (1 + |z|^{p_1} + \dots + |z|^{p_m}) \\ &= |a_0| \rho^{N(r)} |z|^{p_m+1} \left(\frac{1}{|z|} + \dots + \frac{1}{|z|^{p_m+1-p_2}} + \frac{1}{|z|^{p_m+1-p_1}} + \frac{1}{|z|^{p_m+1}} \right). \end{aligned} \quad (14)$$

Using (13) and (14), we have for $|z| \neq 0$

$$\begin{aligned}
 & |R(z)| \\
 & \geq |a_{N(r)}| |z|^{N(r)} - |a_0| \rho^{N(r)} |z|^{p_m+1} \left(\frac{1}{|z|} + \dots + \frac{1}{|z|^{p_m+1-p_1}} + \frac{1}{|z|^{p_m+1}} \right) \\
 & > |a_{N(r)}| |z|^{N(r)} - |a_0| \rho^{N(r)} |z|^{p_m+1} \left(\frac{1}{|z|} + \dots + \frac{1}{|z|^{p_m+1-p_1}} + \frac{1}{|z|^{p_m+1}} + \dots \right) \\
 & = |a_{N(r)}| |z|^{N(r)} - |a_0| \rho^{N(r)} |z|^{p_m+1} \sum_{k=1}^{\infty} \frac{1}{|z|^k}. \tag{15}
 \end{aligned}$$

The geometric series $\sum_{k=1}^{\infty} \frac{1}{|z|^k}$ is convergent for

$$\begin{aligned}
 & \frac{1}{|z|} < 1 \\
 & \text{i.e., for } |z| > 1
 \end{aligned}$$

and converges to

$$\frac{1}{|z|} \frac{1}{1 - \frac{1}{|z|}} = \frac{1}{|z| - 1}.$$

Therefore

$$\sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z| - 1} \text{ for } |z| > 1.$$

So on $|z| > 1$,

$$\begin{aligned}
 & |R(z)| > 0 \text{ if } |a_{N(r)}| |z|^{N(r)} - \frac{|a_0| \rho^{N(r)} |z|^{p_m+1}}{|z| - 1} \geq 0 \\
 & \text{i.e., if } |a_{N(r)}| |z|^{N(r)} \geq \frac{|a_0| \rho^{N(r)} |z|^{p_m+1}}{|z| - 1} \\
 & \text{i.e., if } |a_{N(r)}| |z|^{N(r)+1} - |a_{N(r)}| |z|^{N(r)} \geq |a_0| \rho^{N(r)} |z|^{p_m+1} \\
 & \text{i.e., if } |z|^{p_m+1} \left(|a_{N(r)}| |z|^{N(r)-p_m} - |a_{N(r)}| |z|^{N(r)-p_m-1} - |a_0| \rho^{N(r)} \right) \geq 0.
 \end{aligned}$$

Let us consider

$$g(t) \equiv |a_{N(r)}| |t|^{N(r)-p_m} - |a_{N(r)}| |t|^{N(r)-p_m-1} - |a_0| \rho^{N(r)} = 0.$$

Clearly $g(t) = 0$ has one positive root because the maximum number of changes in sign in $g(t)$ is one and $g(0) = -|a_0| \rho^{N(r)}$ is $-ve$, $g(\infty)$ is $+ve$.

Let t_0 be the positive root of $g(t) = 0$ and $t_0 > 1$. Clearly for $t > t_0$, $g(t) \geq 0$.
If not for some $t_1 > t_0$, $g(t_1) < 0$.

Then $g(t_1) < 0$ and $g(\infty) > 0$. Therefore $g(t) = 0$ must have another positive root in (t_1, ∞) which gives a contradiction .

Hence for $t \geq t_0$, $g(t) \geq 0$ and $t_0 > 1$. So $|R(z)| > 0$ for $|z| \geq t_0$.

Thus $R(z)$ does not vanish in $|z| \geq t_0$.

Hence all the zeros of $R(z)$ lie in $|z| < t_0$.

Let $z = z_0$ be any zero of $P(z)$. So $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$.

Putting $z = \rho z_0$ in $R(z)$ we have

$$R(\rho z_0) = \rho^{N(r)} P(z_0) = \rho^{N(r)} . 0 = 0.$$

Therefore $R(\rho z_0) = 0$ and so $z = \rho z_0$ is a zero of $R(z)$ and consequently $|\rho z_0| < t_0$ which implies $|z_0| < \frac{t_0}{\rho}$. As z_0 is an arbitrary zero of $P(z)$,

$$\text{all the zeros of } P(z) \text{ lie in } |z| < \frac{t_0}{\rho}. \quad (16)$$

Again let us consider

$$F(z) = \rho^{N(r)} z^{N(r)} P\left(\frac{1}{\rho z}\right).$$

Now

$$\begin{aligned} & F(z) \\ = & \rho^{N(r)} z^{N(r)} \left\{ a_0 + a_{p_1} \frac{1}{\rho^{p_1} z^{p_1}} + \dots + a_{p_m} \frac{1}{\rho^{p_m} z^{p_m}} + a_{N(r)} \frac{1}{\rho^{N(r)} z^{N(r)}} \right\} \\ = & a_0 \rho^{N(r)} z^{N(r)} + a_{p_1} \rho^{N(r)-p_1} z^{N(r)-p_1} + \dots + a_{p_m} \rho^{N(r)-p_m} z^{N(r)-p_m} + a_{N(r)}. \end{aligned}$$

Also

$$\begin{aligned} & \left| a_{p_1} \rho^{N(r)-p_1} z^{N(r)-p_1} + \dots + a_{p_m} \rho^{N(r)-p_m} z^{N(r)-p_m} + a_{N(r)} \right| \\ & \leq |a_{p_1}| \rho^{N(r)-p_1} |z|^{N(r)-p_1} + \dots + |a_{p_m}| \rho^{N(r)-p_m} |z|^{N(r)-p_m} + |a_{N(r)}| \\ & \leq |a_{p_1}| \rho^{N(r)-p_1} \left(|z|^{N(r)-p_1} + |z|^{N(r)-p_2} + \dots + |z|^{N(r)-p_m} + 1 \right). \end{aligned}$$

So for $|z| \neq 0$,

$$\begin{aligned}
& |F(z)| \\
& \geq |a_0| \rho^{N(r)} |z|^{N(r)} - |a_{p_1} \rho^{N(r)-p_1} z^{N(r)-p_1} + \dots + a_{p_m} \rho^{N(r)-p_m} z^{N(r)-p_m} + a_{N(r)}| \\
& \geq |a_0| \rho^{N(r)} |z|^{N(r)} - |a_{p_1}| \rho^{N(r)-p_1} \left(|z|^{N(r)-p_1} + |z|^{N(r)-p_2} + \dots + |z|^{N(r)-p_m} + 1 \right) \\
& = |a_0| \rho^{N(r)} |z|^{N(r)} \\
& \quad - |a_{p_1}| \rho^{N(r)-p_1} |z|^{N(r)-p_1+1} \left(\frac{1}{|z|} + \frac{1}{|z|^{p_2-p_1+1}} + \dots + \frac{1}{|z|^{N(r)-p_1+1}} \right)
\end{aligned}$$

i.e., on $|z| \neq 0$,

$$|F(z)| > |a_0| \rho^{N(r)} |z|^{N(r)} - |a_{p_1}| \rho^{N(r)-p_1} |z|^{N(r)-p_1+1} \left(\sum_{k=1}^{\infty} \frac{1}{|z|^k} \right).$$

The geometric series $\sum_{k=1}^{\infty} \frac{1}{|z|^k}$ is convergent for

$$\begin{aligned}
& \frac{1}{|z|} < 1 \\
& \text{i.e., for } |z| > 1
\end{aligned}$$

and converges to

$$\frac{1}{|z|} \frac{1}{1 - \frac{1}{|z|}} = \frac{1}{|z| - 1}.$$

Therefore

$$\sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z| - 1} \text{ for } |z| > 1.$$

Therefore for $|z| > 1$

$$\begin{aligned}
|F(z)| & > |a_0| \rho^{N(r)} |z|^{N(r)} - |a_{p_1}| \rho^{N(r)-p_1} |z|^{N(r)-p_1+1} \left(\frac{1}{|z| - 1} \right) \\
& = \rho^{N(r)-p_1} \left(\rho^{p_1} |a_0| |z|^{N(r)} - |a_{p_1}| \frac{|z|^{N(r)-p_1+1}}{|z| - 1} \right) \\
& = \rho^{N(r)-p_1} |z|^{N(r)-p_1+1} \left(|a_0| \rho^{p_1} |z|^{p_1-1} - \frac{|a_{p_1}|}{|z| - 1} \right)
\end{aligned}$$

For $|z| > 1$,

$$\begin{aligned}
 |F(z)| > 0 \text{ if } |a_0| \rho^{p_1} |z|^{p_1-1} - \frac{|a_{p_1}|}{|z|-1} &\geq 0 \\
 \text{i.e., if } |a_0| \rho^{p_1} |z|^{p_1-1} &\geq \frac{|a_{p_1}|}{|z|-1} \\
 \text{i.e., if } |a_0| \rho^{p_1} |z|^{p_1} - |a_0| \rho^{p_1} |z|^{p_1-1} - |a_{p_1}| &\geq 0. \quad (17)
 \end{aligned}$$

Therefore on $|z| > 1$, $|F(z)| > 0$ if (17) holds.

Let us consider

$$f(t) = |a_0| \rho^{p_1} t^{p_1} - |a_0| \rho^{p_1} t^{p_1-1} - |a_{p_1}| = 0.$$

Clearly $f(t) = 0$ has exactly one positive root and is greater than one. Let t'_0 be the positive root of $f(t) = 0$. Therefore $t'_0 > 1$. Obviously if $t \geq t'_0$ then $f(t) \geq 0$. So for $|F(z)| > 0$, $|z| \geq t'_0$. Therefore $F(z)$ does not vanish in $|z| \geq t'_0$. Hence all the zeros of $F(z)$ lie in $|z| < t'_0$.

Let $z = z_0$ be any zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$. Now putting $z = \frac{1}{\rho z_0}$ in $F(z)$ we obtain that

$$\begin{aligned}
 F\left(\frac{1}{\rho z_0}\right) &= \rho^{N(r)} \left(\frac{1}{\rho z_0}\right)^{N(r)} P(z_0) \\
 &= \left(\frac{1}{z_0}\right)^{N(r)} P(z_0) = 0.
 \end{aligned}$$

Therefore $z = \frac{1}{\rho z_0}$ is a zero of $F(z)$. Now

$$\begin{aligned}
 \left|\frac{1}{\rho z_0}\right| &< t'_0 \\
 \text{i.e., } \left|\frac{1}{z_0}\right| &< \rho t'_0 \\
 \text{i.e., } |z_0| &> \frac{1}{\rho t'_0}.
 \end{aligned}$$

As z_0 is an arbitrary zero of $P(z)$ therefore we obtain that

$$\text{all the zeros of } P(z) \text{ lie in } |z| > \frac{1}{\rho t'_0}. \quad (18)$$

Using (16) and (18) we get that all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{\rho t'_0} < |z| < \frac{t_0}{\rho}$$

where t_0, t'_0 are the unique positive roots of the equations $g(t) = 0$ and $f(t) = 0$ respectively whose form is given in the statement of Theorem 3. ■

Corollary 6 *In view of Theorem 3 we may state that all the zeros of the polynomial $P(z) = a_0 + a_{p_1}z^{p_1} + \dots + a_{p_m}z^{p_m} + a_nz^n$ of degree n with $1 \leq p_1 < p_2 < \dots < p_m \leq n-1, p_i$'s are integers such that*

$$|a_0| \geq |a_{p_1}| \geq \dots \geq |a_n|$$

lie in ring shaped region

$$\frac{1}{t'_0} < |z| < t_0$$

where t_0, t'_0 are the unique positive roots of the equations

$$g(t) \equiv |a_n|t^{n-p_m} - |a_n|t^{n-p_m-1} - |a_0| = 0$$

and

$$f(t) \equiv |a_0|t^{p_1} - |a_0|t^{p_1-1} - |a_{p_1}| = 0$$

respectively just substituting $\rho = 1$.

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