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Fixed Point Theorems for ϕ-contractive Type Mappings in Dislocated Quasi- Metric Spaces.

Kastriot Zoto

Department of Mathematics and Computer Sciences
Faculty of Natural Sciences; University of Gjirokastra
Gjirokastra, Albania
zotokastriot@yahoo.com

Elida Hoxha

Department of Mathematics; Faculty of Natural Sciences
University of Tirana;Tirana, Albania
hoxhaelida@yahoo.com

Abstract

In this paper we establish fixed point theorems for ϕ-contractive mappings in complete dislocated quasi- metric spaces. Our theorems extend and generalize results in literature for dq metric spaces.

Mathematics Subject Classification:

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1 Introduction

The Banach contraction principle is the basic result in fixed point theory. It has many applications in various branches of mathematics .Since then, many authors have been studying many contractions and proved fixed point theorems. P. Hitzler [5] introduced the concept of dislocated metric space as a generalization of metric spaces and presented variants of Banach' Contraction Principle in dislocated metric spaces.In 2005, Zeyada, F.M. et al [3] introduced the notion of dislocated quasi-metric space and generalized the result of Hitzler in such spaces.In this paper we introduce the concept of ϕ-contractive mapping and establish fixed point theorems in dislocated quasi- metric spaces, which generalize and unify the results of above authors [5] and[3]. Our theorems obtain fixed point results omitting some conditions of mapping.

2 Preliminaries

We start with base and auxiliary definitions and notations, which will be used throughout in this paper.

Definition 2.1 [3] Let X be a nonempty set and let $d: X \times X \to [0,\infty)$ be a function satisfying following conditions:
$d_1: d(x,y) = d(y,x) = 0 \Rightarrow x = y$
$d_2: d(x,y) \le d(x,z) + d(z,y)$ *for all* $x,y,z \in X$
Then d is called a dislocated quasi-metric on X. If d satisfies $d(x,x) = 0$, for all x $\in$ X, then the dislocated quasi-metric is called a quasi-metric on X. If d satisfies $d(x,y) = d(y,x)$, for all $x,y \in X$ then the dislocated quasi-metric is called a dislocated metric on X.

Definition 2.2 [3] A sequence $(x_n)_{n\in\mathbb{N}}$ in dq-metric space (X,d) is called Cauchy if for all $\varepsilon > 0$, $\exists n_0 \in N$ such that $\forall m,n \ge n_0$, $d(x_m,x_n) < \varepsilon$ or $d(x_n,x_m) < \varepsilon$.

Definition 2.3[3]A sequence $(x_n)_{n\in\mathbb{N}}$ dislocated quasi converges or dq-converges to x if $\lim_{n\to\infty} d(x_n,x) = \lim_{n\to\infty} d(x,x_n) = 0$. In this case x in called a dq-limit of $(x_n)_{n\in\mathbb{N}}$ and we write $x_n \to x$.

Definition 2.4 [3] A dq-metric space (X,d) is complete, if every Cauchy sequence in it is dq-convergent.

Lemma 2.5 [3] Every subsequence of dq-convergent sequence to a point x_0 is dq-convergent to x_0.

Definition 2.6 [3] Let (X,d) be a dq-metric space. A mapping $T: X \to X$ is called contraction if there exists $0 \le \lambda < 1$ such that:
$d(Tx,Ty) \le \lambda d(x,y)$ for all $x,y \in X$.

Lemma 2.7 [3] Let (X,d) be a dq-metric space. If $f: X \to X$ is a contraction function, then $f^n(x_0)$ is a Cauchy sequence for each $x_0 \in X$.

Lemma 2.8 [3] dq-limits in a dq-metric space are unique.

Theorem 2.9 [3] Let (X,d) be complete dq-metric space and let $T: X \to X$ be a continuous contraction function, then T has a unique fixed point.

3 Main results

Denote with Φ the family of non-decreasing functions: $\phi:[0,+\infty)\to[0,+\infty)$ such that $\sum_{n=1}^{+\infty}\phi^n(t)<+\infty$ for each $t>0$, where ϕ^n is the n-th iterate of ϕ.

Lemma 3.1 For every function $\phi:[0,+\infty)\to[0,+\infty)$ holds the following:
If ϕ is non-decreasing, then for each $t>0$, $\lim_{n\to\infty}\phi^n(t)=0$ implies $\phi(t)<t$.

Definition 3.2 Let (X,d) be a complete dislocated quasi- metric space and $T: X\to X$ be a given mapping. We say that T is a ϕ-contractive mapping, if there exists a function $\phi\in\Phi$ such that:

$$d(Tx.Ty)\le\phi(d(x,y)) \tag{1}$$

for all $x,y\in X$.

Definition 3.3 Let (X,d) be a dislocated quasi- metric space and $T: X\to X$ be a given mapping **.**For each $x\in X$, let $\mathrm{O}(x)=\{x,Tx,T^2x,\ldots\}$ which will be called the orbit of T at x. (X,d) is called T-orbitally complete if and only if every Cauchy sequence which is contained in $\mathrm{O}(x)$ dislocated converges to a point in X .

Theorem 3.4 Let (X,d) be a complete dislocated quasi- metric space and $T: X\to X$ be a continuous ϕ-contractive mapping. Then, T has a fixed point in X .

Proof. Let x_0 be an arbitrary point in X .Define the sequence $(x_n)_{n\in\mathbb{N}}$ in X as follows: $x_{n+1}=Tx_n$, for all $n\ge 0$.
We consider these cases:
Case 1. If $x_n=x_{n+1}$ for some $n\in\mathbb{N}$, then we have $Tx_n=x_{n+1}$ so $u=x_n$ is a fixed point of T .
Case 2. Assume that $x_n\ne x_{n+1}$ for all $n\in\mathbb{N}$. By the condition (1) we have:

$$d(x_n,x_{n+1})=d(Tx_{n-1},Tx_n)\le\phi(d(x_{n-1},x_n))$$

Similarly:

$$d\left(x_{n-1},x_n\right)=d\left(Tx_{n-2},Tx_{n-1}\right)\le\phi\Big(d\left(x_{n-2},x_{n-1}\right)\Big)$$

In this way by induction, we get

$$d\left(x_n,x_{n+1}\right)\le\phi^n\left(d\left(x_0,x_1\right)\right)\quad\text{for all}\quad n\in\mathbb{N}.$$

Let $\varepsilon>0$ (fix) and $n(\varepsilon)\in\mathbb{N}$, such that $\sum_{n>n(\varepsilon)}\phi^n\left(d\left(x_0,x_1\right)\right)<\varepsilon$.

Let $n,m\in\mathbb{N}$ with $m>n>n(\varepsilon)$, using the triangular inequality, we obtain:

$$\begin{aligned}d\left(x_n,x_m\right)&\le d\left(x_n,x_{n+1}\right)+d\left(x_{n+1},x_{n+2}\right)+....+d\left(x_{m-1},x_m\right)\\&\le\phi^n\left(d\left(x_0,x_1\right)\right)+\phi^{n+1}\left(d\left(x_0,x_1\right)\right)+....+\phi^{m-1}\left(d\left(x_0,x_1\right)\right)\\&=\sum_{k=n}^{m-1}\phi^k\left(d\left(x_0,x_1\right)\right)\\&\le\sum_{n>n(\varepsilon)}\phi^n\left(d\left(x_0,x_1\right)\right)<\varepsilon\end{aligned}$$

Similarly: $d\left(x_m,x_n\right)<\varepsilon$.

Thus we have proved that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete dislocated quasi-metric space (X,d). Since X is complete, there exists $u\in X$, such that $x_n\to u$ (dislocated quasi converges) as $n\to\infty$.

From the continuity of T, we have:

$$u=\lim_{n\to\infty}\left(x_{n+1}\right)=\lim_{n\to\infty}\left(Tx_n\right)=T\left(\lim_{n\to\infty}\left(x_n\right)\right)=Tu.$$

Thus u is a fixed point of T.

Uniqueness. Suppose that u and v are two fixed point of T.From triangular inequality and condition (1) we have:

$$\begin{aligned}d\left(u,v\right)&\le d\left(u,x_{n+1}\right)+d\left(x_{n+1},v\right)\\&=d\left(Tu,T^n\left(x_0\right)\right)+d\left(T^n\left(x_0\right),Tv\right)\\&=d\left(Tu,T\left(T^{n-1}\left(x_0\right)\right)\right)+d\left(T\left(T^{n-1}\left(x_0\right)\right),Tv\right)\\&\le\phi\Big(d\left(u,T^{n-1}\left(x_0\right)\right)\Big)+\phi\Big(d\left(T^{n-1}\left(x_0\right),v\right)\Big)\end{aligned}$$

Continuing in this way we have

$$d\left(u,v\right)\le\phi^n\left(d\left(u,x_0\right)\right)+\phi^n\left(d\left(x_0,v\right)\right)\quad\text{for all}\quad n\in\mathbb{N}$$

Letting $n\to\infty$ we have $d\left(u,v\right)=0$. Similarly, we get $d\left(v,u\right)=0$.

Therefore: $d\left(u,v\right)=d\left(v,u\right)=0$ implies $u=v$. Hence fixed point is unique.

The following example illustrates theorem 3.4

Example 3.5 $X=\{1,2,3\}$ and $d\left(x,y\right)=\max\left\{x,y^2\right\}$.

Define $T:X\to X$, as $T\left(1\right)=T\left(2\right)=T\left(3\right)=2$. T is continuous and

$d(Tx,Ty) \le \phi(d(x,y))$ for all $x,y \in X$, where $\phi(t) = \frac{t}{4}$. Clearly 2 is a unique fixed point of T.

In the following theorem we omit the continuity of T.

Theorem3.6 Let (X,d) be a complete dislocated quasi-metric space and $T : X \to X$ be a ϕ-contractive mapping. Then, T has a fixed point in X.

Proof. In the same way as in theorem 3.4 we can show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete dislocated quasi-metric space (X,d). Then, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. Using the triangular inequality and condition (1), we get

$$\begin{aligned} d(Tu,u) &\le d(Tu,Tx_n) + d(Tx_n,u) \\ &\le \phi(d(u,x_n)) + d(x_{n+1},u) \end{aligned}$$

Letting $n \to \infty$ since ϕ is continuous at $t = 0$, we obtain $d(Tu,u) = 0$.

Similarly, by the same argument, we obtain $d(u,Tu) = 0$

Therefore $d(Tu,u) = d(u,Tu) = 0$, which implies $Tu = u$. Hence u is a fixed point of T.

Uniqueness. Let v be another fixed point of T.

By the triangular inequality and (1), we have:

$$\begin{aligned} d(u,v) &= d(Tu,Tv) \\ &\le d(Tu,Tx_n) + d(Tx_n,Tv) \\ &\le \phi^n(d(u,x_0)) + \phi^n(d(x_0,v)) \end{aligned}$$

Letting $n \to \infty$ we have $d(u,v) = 0$. Similarly $d(v,u) = 0$

Therefore: $d(u,v) = d(v,u) = 0$ implies $u = v$. Hence fixed point is unique.

Theorem 3.7 Let (X,d) be a dislocated quasi- metric space and T-orbitally complete, and $T : X \to X$ be a continuous ϕ-contractive mapping. Then, T has a fixed point in X.

Proof. Similarly as in the theorems 3.4 and 3.6, we know that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in dislocated quasi-metric space (X,d). Since (X,d) is T-orbitally complete then, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$.

By the continuity of T, we have :

$$u = \lim_{n\to\infty}(x_{n+1}) = \lim_{n\to\infty}(Tx_n) = T\left(\lim_{n\to\infty}(x_n)\right) = Tu$$

Thus u is a fixed point of T. In the same way as in theorem above, we prove that fixed point is unique.

Theorem 3.8 Let (X,d) be a dislocated quasi- metric space and T-orbitally complete, and $T:X\to X$ be a ϕ-contractive mapping. Then, T has a fixed point in X **.**

Proof. In the same way as in theorems above, we prove existence and uniqueness of the fixed point.

Remark 3.9 If we define the function $\phi:\phi(t)=kt$ for all $t\geq 0$ and some $k\in[0,1)$ is taken theorem 2.1 of F.M. Zeyada [3].

4 Conclusion

In this paper we prove existence and uniqueness of fixed point, for any self-mapping $T:X\to X$ which is ϕ-contractive (satisfies condition (1)).

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