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# On Some Common Fixed Point Theorems in Modulared Spaces

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**Abstract**

Fixed point theorems have important roles and applications in many areas, particularly in differential and integral equations. One most popular fixed point theorem is those so-called Banach fixed point theorem. Some authors have generalized the concept of Banach fixed point into modulared spaces. In this paper, we present some conditions that guarantee the existence of the common fixed point for a pairs of operators on a modulared space.

**Mathematics Subject Classification:** 46A40, 46B40, 47H10

**Keywords:** common fixed point theorems, modulared spaces

## 1 Introduction

One knows that fixed point theorems have important roles and applications in many areas, particularly in differential and integral equations [1]. One most popular fixed point theorem is those so-called Banach fixed point theorem. The theory of Banach fixed point discussed in [2].

The theory of modulared spaces was initiated by Nakano [6]. Some properties and characterization of the spaces are discussed in [3] and [6]. Geometrically, many concept in normed spaces closed relate to those in modulared spaces. Following the fact, some authors, among others are Kumam, P. in [4] and Marzouki, B. in [5], can generalize the concept of Banach fixed point theorem into some classes of modulared spaces.

To this end, we will present some conditions that guarantee the existence of the common fixed point for a pairs of operators on a modulared space.

## 2 A Modulared Space and Its Basic Properties

As usual, $\mathcal{N}$ and $\mathcal{R}$ denote the set of all positive integers and real numbers system, respectively. The extended real numbers system will be denoted by $\mathcal{R}^*$.

Let $X$ be a linear space over $\mathcal{R}$. A non-negative function $\rho : X \to \mathcal{R}^*$ is called a *modular* if for every $f, g \in X$ the following conditions hold,

(i) $\rho(f) = 0$ iff $f = 0$.

(ii) $\rho(-f) = \rho(f)$.

(iii) $\rho(\alpha f \vee \beta g) \leq \rho(f) + \rho(g)$ for every $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$.

If we change the axiom (iii) by

(iii') $\rho(\alpha f \vee \beta g) \leq \alpha\rho(f) + \alpha\rho(g)$ for every $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$,

then we say that the modular $\rho$ is a modular convex. A linear space $X$ equipped with a modular $\rho$, written by $(X, \rho)$, is called a modulared space. We shall also denote a modulared space by the single character $X$, when the modular $\rho$ is explicitly understood.

By considering the definition of the modular, then we can easily prove the following theorems and corollary.

**Theorem 2.1** *Let $(X, \rho)$ be a modulared space.*

(i) *If $\alpha, \beta \in \mathcal{R}$, $0 \leq \alpha \leq \beta$ then $\rho(\alpha f) \leq \rho(\beta f)$ for every $f \in X$.*

(ii) *If $\rho(f) < \epsilon$ for every $\epsilon > 0$ then $f = 0$.*

**Theorem 2.2** *Let $(X, \rho)$ be a modulared space. If $f_1, f_2, f_3, \ldots, f_n \in X$, and $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are non-negative real numbers such that $\sum_{i=1}^n \alpha_i = 1$, then $\rho(\sum_{i=1}^n \alpha_i f_i) \leq \sum_{i=1}^n \rho((f_i)$.*

**Corollary 2.3** *Let $(X, \rho)$ be a modulared space, where the modular $\rho$ is convex. If $f_1, f_2, f_3, \ldots, f_n \in X$, and $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are non-negative real numbers such that $\sum_{i=1}^n \alpha_i = 1$, then $\rho(\sum_{i=1}^n \alpha_i f_i) = \sum_{i=1}^n \alpha_i \rho((f_i)$.*

*Trough out this paper we always assume that the modular $\rho$ is always convex, unless otherwise stated.*

Let $(X, \rho)$ be a modulared space. In this paper, by the modulared space, we always mean a linear subspace $\mathcal{M} \subset X$ such that

$$\mathcal{M} = \{f \in X : \quad \rho(f) < \infty\}.$$

Let $(\mathcal{M}, \rho)$ be a modulared space. A sequence $\{f^{(n)}\} \subset \mathcal{M}$ is said to be *$\rho$-convergent* (*modular convergent*) to $f \in \mathcal{M}$ if for every real number $\epsilon > 0$ there exists a positive integer $N$ such that for every $n \geq N$ we have:

$$\rho((f^{(n)} - f)) < \epsilon.$$

Further, $f$ is called a *modular limit* (*$\rho$-limit*) of $\{f^{(n)}\}$, and we write

$$\rho - \lim_{n \to \infty} f^{(n)} = f$$

If the sequence $\{f^{(n)}\} \subset \mathcal{M}$ is $\rho$-convergent, then we can prove that its $\rho$-limit is unique. A sequence $\{f^{(n)}\} \subset \mathcal{M}$ is called a *$\rho$-Cauchy* (*modular Cauchy*) sequence if for every real number $\epsilon > 0$ there exists a positive integer $N$ such that for every $m, n \geq N$ we have:

$$\rho(f^{(n)} - f^{m}) < \epsilon.$$

It is clear that in every modulared space, every $\rho$-convergent sequence is $\rho$-Cauchy sequence. However, the converse may be failed. The modulared space $\mathcal{M}$ is said to be $\rho$-complete if every $\rho$-Cauchy sequence in $\mathcal{M}$ is $\rho$-convergent.

**Definition 2.4** *Let $\mathcal{M}$ be a modulared space. Any set $F \subset \mathcal{M}$ is said to be modular close ($\rho$-closed) if for any sequence $\{f_n\} \subset F$ which is $\rho$-convergent to $f \in \mathcal{M}$ implies $f \in F$.*

**Definition 2.5** *Let $\mathcal{M}$ be a modulared space. Any set $B \subset \mathcal{M}$ is said to be modular bounded ($\rho$-bounded) if* $\sup\{\rho(f - g) : \ f,\ g \in B\} < \infty$.

# 3 Fixed Point Theorems on Modulared Spaces

We begin this section by proving the following theorem.

**Theorem 3.1** *Let $(\mathcal{M}, \rho)$ be a $\rho$-complete modulared space and $B \subset \mathcal{M}$ $\rho$-closed and $\rho$-bounded set. If the operators $S, T : B \to B$ satisfy $ST = TS$ and*

$$\rho(2(T(f) - T(g))) \leq k\rho(S(f) - S(g))$$

*for every $f, g \in B$ and for some $k \in (0, 2)$, then $S$ and $T$ have a unique common fixed point.*

**Proof.** Let $f_0 \in B$. For any integer $n \geq 0$, we define

$$S(f_{n+1}) = T(f_n)$$

Let $\alpha_0 = \rho(S(f_0))$ and $\alpha_{n+1} = \rho(S(f_{n+1}) - S(f_n))$ for every $n \in \mathcal{N} \cup \{0\}$. Since $\rho$ is convex, then for any $n \in \mathcal{N}$, we have

$$\begin{aligned}
\alpha_{n+1} &= \rho(T(f_n) - T(f_{n-1})) \\
&= \rho(\frac{1}{2}2(T(f_n) - T(f_{n-1}))) \leq \frac{1}{2}\rho(2(T(f_n) - T(f_{n-1}))) \\
&\leq \frac{1}{2}k\rho(S(f_n) - S(f_{n-1})) = \frac{k}{2}\alpha_n
\end{aligned}$$

As the consequence, then we have $\alpha_{n+1} \leq \frac{k}{2}^{n+1}\alpha_0$. Further, for any integer $p \geq 2$, we have

$$\begin{aligned}
\rho(S(f_{n+p}) - S(f_n)) &= \rho(T(f_{n+p-1}) - T(f_{n-1})) \\
&= \rho(\frac{1}{2}2(T(f_{n+p-1}) - T(f_{n+p-2})) + \frac{1}{2}2(T(f_{n+p-2}) - T(f_{n-1}))) \\
&\leq \frac{k}{2}\rho(S(f_{n+p-1}) - S(f_{n+p-2})) + \frac{k}{2}\rho(S(f_{n+p-2}) - S(f_{n-1})) \\
&\leq \frac{k}{2}\alpha_{n+p-1} \\
&\quad + \frac{k}{2}\{\rho(\frac{1}{2}2(T(f_{n+p-3}) - T(f_{n+p-4})) + \frac{1}{2}2(T(f_{n+p-4} - f_{n-2})))\} \\
&\leq \frac{k}{2}\alpha_{n+p-1} + (\frac{k}{2})^2\alpha_{n+p-3} + (\frac{k}{2})^2\rho(S(f_{n+p-4}) - S(f_{n-2})) \\
&\leq \ldots \\
&\leq \{(\frac{k}{2})^{n+p} + (\frac{k}{2})^{n+p-1} + \ldots + (\frac{k}{2})^{n-1}\}(\alpha_0) \to 0,
\end{aligned}$$

as $n \to \infty$. This implies that $\{S(f_n)\}$ and $\{T(f_n)\}$ are $\rho$-Cauchy sequences. Following the $\rho$-completeness of $\mathcal{M}$, then $\{S(f_n)\}$ and $\{T(f_n)\}$ are $\rho$-convergence in $\mathcal{M}$ and, by considering the construction of $\{f_n\}$, then we have

$$\lim_{n\to\infty} T(f_n) = \lim_{n\to\infty} S(f_n) = g$$

for some $g \in \mathcal{M}$. Since, $B$ is $\rho$-closed, then $g \in B$. Further, there exists an $f \in B$ such that

$$T(f) = g = S(f)$$

It implies that $T(T(f)) = S(S(f))$, since $ST = TS$.

Further, by proving $\rho(T(f) - T^2(f)) = 0$, then we have

$$T(f) = T(T(f)) = S(T(f))$$

i.e. $T(f) = g \in B$ is a common fixed point of $T$ and $S$. The uniqueness of the fixed point follows from the convexity of the modular $\rho$. □

Other conditions that also guarantee the existence of the common fixed point of any two operators are presented in the next theorems.

For any function $\psi : D \subset \mathcal{R}^+ \to \mathcal{R}^+$ and $n \in \mathcal{N}$, we define a function $\psi^n$ as $n$ times composition of $\psi$. A function $\psi : D \subset \mathcal{R}^+ \to \mathcal{R}^+$ is said to be contractive if it is non-decreasing, $\psi(t) > 0$ for every $t > 0$, and $\lim_{n\to\infty} \psi^n(t) = 0$ for every $t \geq 0$.

The following lemma is needed to prove the next theorem.

**Lemma 3.2** *If $\psi : D \subset \mathcal{R}^+ \to \mathcal{R}^+$ is a contractive function, then $\psi(t) < t$ for every $t > 0$.*

**Proof.** Suppose the contrary is true, i.e. $\psi(t) \geq t$ for some $t > 0$, then for any positive integer $n$, we have

$$\psi^n(t) \geq t$$

This contradicts to the fact that $\lim_{n\to\infty} \psi^n(t) = 0$ for every $t \geq 0$. □

**Theorem 3.3** *Let $(\mathcal{M}, \rho)$ be a $\rho$-complete modulared space, where the modular $\rho$ is convex, $B \subset \mathcal{M}$ $\rho$-closed and $\rho$-bounded set, and $\psi : \mathcal{R}^+ \to \mathcal{R}^+$ a contractive function. If the operators $S, T : B \to B$ satisfies $ST = TS$ and*

$$\rho(2(T(f) - T(g))) \leq k\psi(\rho(S(f) - S(g)))$$

*for every $f, g \in B$ and for some $k \in (0, 2)$, then $S$ and $T$ have a unique common fixed point.*

**Proof.** Let $f_0 \in B$. For any integer $n \geq 0$, we define

$$S(f_{n+1}) = T(f_n)$$

Let $\alpha_0 = \rho(S(f_0))$ and $\alpha_{n+1} = \rho(S(f_{n+1}) - S(f_n))$ for every $n \in \mathcal{N} \cup \{0\}$. Since $\rho$ is convex, then for any $n \in \mathcal{N}$, we have

$$\begin{aligned} \alpha_{n+1} &= \rho(T(f_n) - T(f_{n-1})) \\ &= \rho(\frac{1}{2}2(T(f_n) - T(f_{n-1}))) \leq \frac{1}{2}\rho(2(T(f_n) - T(f_{n-1}))) \\ &\leq \frac{1}{2}k\psi(\rho(S(f_n) - S(f_{n-1}))) = \frac{k}{2}\psi(\alpha_n) \end{aligned}$$

As the consequence, since $\psi$ is non decreasing, then we have $\alpha_{n+1} \leq \frac{k}{2}\psi^{n+1}(\alpha_0)$. Further, for any integer $p \geq 2$, we have

$$\begin{aligned}
\rho(S(f_{n+p}) - S(f_n)) &= \rho(T(f_{n+p-1}) - T(f_{n-1})) \\
&= \rho(\frac{1}{2}2(T(f_{n+p-1}) - T(f_{n+p-2})) + \frac{1}{2}2(T(f_{n+p-2}) - T(f_{n-1}))) \\
&\leq \frac{k}{2}\psi(\rho(S(f_{n+p-1}) - S(f_{n+p-2}))) + \frac{k}{2}\psi(\rho(S(f_{n+p-2}) - S(f_{n-1}))) \\
&\leq \frac{k}{2}\psi(\alpha_{n+p-1}) \\
&\quad + \frac{k}{2}\{\rho(\frac{1}{2}2(T(f_{n+p-3}) - T(f_{n+p-4})) + \frac{1}{2}2(T(f_{n+p-4} - f_{n-2})))\} \\
&\leq \frac{k}{2}\psi(\alpha_{n+p-1}) + (\frac{k}{2})^2\psi(\alpha_{n+p-3}) + (\frac{k}{2})^2\psi(\rho(S(f_{n+p-4}) - S(f_{n-2}))) \\
&\leq \ldots \\
&\leq \frac{k}{2}\psi^{n+p}(\alpha_0) + (\frac{k}{2})^2\psi^{n+p-2}(\alpha_0) + \ldots + (\frac{k}{2})^p\psi^n(\alpha_0) \to 0,
\end{aligned}$$

as $n \to \infty$. This implies that $\{S(f_n)\}$ and $\{T(f_n)\}$ are $\rho$-Cauchy sequences. Following the $\rho$-completeness of $\mathcal{M}$, then $\{S(f_n)\}$ and $\{T(f_n)\}$ are $\rho$-convergence in $\mathcal{M}$ and

$$\lim_{n\to\infty} T(f_n) = \lim_{n\to\infty} S(f_n) = g$$

for some $g \in \mathcal{M}$. Since, $B$ is $\rho$-closed, then $g \in B$. Further, there exists an $f \in B$ such that

$$T(f) = g = S(f)$$

It implies that $T(T(f)) = S(S(f))$, since $ST = TS$.

Suppose that $\rho(T(f) - T^2(f)) \neq 0$, then following the Lemma 3.2, we have

$$\begin{aligned}
\rho(T(f) - T^2(f)) &\leq \frac{1}{2}\rho(2(T(f) - T^2(f))) \\
&\leq \frac{k}{2}\psi(\rho(S(f) - S(S(f)))) \\
&< \rho(T(f) - T^2(f))
\end{aligned}$$

However, the above inequality is impossible. So, $\rho(T(f) - T^2(f)) = 0$. As a consequence, we have

$$T(f) = T(T(f)) = S(T(f))$$

i.e. $T(f) = g \in B$ is a common fixed point of $T$ and $S$. The uniqueness of the fixed point follows from the convexity of the modular $\rho$. □

Let $a > 0$ be an arbitrary. A function $\phi : [0, a] \to \mathcal{R}^+$ is said to be expansive if it is non-decreasing, $\phi(t) > t$ for every $t > 0$, and the condition $\lim_{n\to\infty} \phi(t_n) = \lim_{n\to\infty} t_n = t$ implies $t = 0$. Further, we have the following theorem.

**Theorem 3.4** *Let $(\mathcal{M}, \rho)$ be a $\rho$-complete modulared space, $B \subset \mathcal{M}$ $\rho$-closed and $\rho$-bounded set, and $\phi : \mathcal{R}^+ \to \mathcal{R}^+$ an expansive function. If the operators $S, T : B \to B$ satisfies $ST = TS$ and*

$$\phi(\rho(2(T(f) - T(g)))) \leq k\rho(S(f) - S(g))$$

*for every $f, g \in B$ and for some $k \in (0, 2)$, then $S$ and $T$ have a unique common fixed point.*

**Proof.** Let $f_0 \in B$. For any integer $n \geq 0$, we define

$$S(f_{n+1}) = T(f_n)$$

Let $\alpha_0 = \rho(S(f_0))$ and $\alpha_{n+1} = \rho(2(S(f_{n+1}) - S(f_n)))$ for every $n \in \mathcal{N} \cup \{0\}$. Since $\phi$ is expansive and following the convexity of $\rho$, then for any $n \in \mathcal{N}$ we have

$$\begin{aligned} \alpha_{n+1} &\leq \phi(\rho(2(T(f_n) - T(f_{n-1})))) \\ &\leq k\rho(S(f_n) - S(f_{n-1})) \leq \rho(2(S(f_n) - S(f_{n-1}))) = \alpha_n \end{aligned}$$

i.e. $\{\alpha_n\}$ is decreasing. Since, it is bounded below, then $\lim_{n\to\infty} \alpha_n$ exists. Notice that

$$\begin{aligned} \alpha_n &< \phi(\alpha_n) = \phi(\rho(2(T(f_{n-1}) - T(f_{n-2})))) \\ &\leq k\rho(S(f_{n-1}) - S(f_{n-2})) \leq \rho(2(S(f_{n-1}) - S(f_{n-2}))) = \alpha_{n-1} \end{aligned}$$

This implies $\lim_{n\to\infty} \phi(\alpha_n)$ exists and

$$\lim_{n\to\infty} \phi(\alpha_n) = \lim_{n\to\infty} \alpha_n$$

Since $\phi$ is expansive, then

$$\lim_{n\to\infty} \phi(\alpha_n) = \lim_{n\to\infty} \alpha_n = 0$$

Hence,

$$\lim_{n\to\infty} \rho(2(T(f_n) - S(f_{n-1}))) = \lim_{n\to\infty} \rho(2(S(f_{n+1}) - S(f_n))) = 0$$

This implies that $\{S(f_n)\}$ and $\{T(f_n)\}$ are $\rho$-Cauchy sequences. Following the $\rho$-completeness of $\mathcal{M}$, then $\{S(f_n)\}$ and $\{T(f_n)\}$ are $\rho$-convergence in $\mathcal{M}$ and

$$\lim_{n\to\infty} T(f_n) = \lim_{n\to\infty} S(f_n) = g$$

for some $g \in \mathcal{M}$. Since, $B$ is $\rho$-closed, then $g \in B$. Further, there exists an $f \in B$ such that

$$T(f) = g = S(f)$$

It implies that $T(T(f)) = S(S(f))$, since $ST = TS$.

Suppose that $\rho(T(f) - T^2(f)) \neq 0$, then following the Lemma 3.2, we have

$$\begin{aligned} \rho(T(f) - T^2(f)) &\leq \frac{1}{2}\rho(2(T(f) - T^2(f))) \\ &\leq \frac{k}{2}\psi(\rho(S(f) - S(S(f)))) \\ &< \rho(T(f) - T^2(f)) \end{aligned}$$

However, the above inequality is impossible. So, $\rho(T(f) - T^2(f)) = 0$. As a consequence, we have

$$T(f) = T(T(f)) = S(T(f))$$

i.e. $T(f) = g \in B$ is a common fixed point of $T$ and $S$. The uniqueness of the fixed point follows from the convexity of the modular $\rho$. □

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