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Upper and Lower $\hat{\omega}$-Continuous Multifunctions

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Abstract

The aim of this paper is to introduce a new class of continuous multifunctions, namely upper and lower $\hat{\omega}$-continuous multifunctions and to obtain some characterizations concerning upper and lower $\hat{\omega}$-continuous multifunctions.

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1 Introduction

One of the important and basic topics in the theory of classical point set topology and in several branches of mathematics,which has been investigated by many authors is continuity of functions.This concept has been extended to the setting of multifunctions.The aim of this paper is to introduce upper (lower) $\hat{\omega}$-continuous multifunctions and to obtain several characterizations of such multifunctions.

2 Preliminaries

Throughout this paper (X, τ) and (Y, σ) and (Z, η)(or simply X,Y and Z represent non-empty topological spaces.For a subset A of (X, τ),cl(A),int(A) and A^c denote the closure of A,the interior of A and the complement of A respectively.A subset A of a space (X, τ) is called θ-closed[11] if $A = cl_\theta(A)$,where $cl_\theta(A) = \{x \in X : cl(U) \cap A \neq \phi, U \in \tau$ and $x \in U\}$.The complement of θ-closed set is called θ-open set.A subset A of a space $(X, \tau)$is called δ-closed [11] if $A = cl_\delta(A)$,where $cl_\delta(A) = \{x \in X : int(cl(U)) \cap A \neq \phi, U \in \tau$ and $x \in U\}$.The complement of δ-closed set is called δ-open set.

Definition 2.1 *A subset A of a space X is called an a-open set [3] if A* $\subset$ *int(cl(int*$_\delta$*(A)).The complement of a-open set is called a-closed set.The a-closure of a subset A of X is the intersection of all the a-closed sets containing A and it is denoted by acl(A).*

Definition 2.2 *A subset A of a space* (X, τ) *is called a*

(i)$\hat{g}$*-closed set[10] if cl(A)*$\subseteq$*U whenever A*$\subseteq$*U and U is semi-open in* (X, τ).

(ii)$\alpha\hat{g}$*-closed set [2] if* $\alpha cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and U is* $\hat{g}$*-open in* (X, τ).

(iii)$\hat{\omega}$*-closed set [6] if acl(A)*$\subseteq$ *U whenever* $A \subseteq U$ *and U is* $\alpha\hat{g}$*-open in* (X, τ).

The complement of $\hat{g}$*-closed(*$\alpha\hat{g}$*-closed and* $\hat{\omega}$*-closed)set is called* $\hat{g}$*-open (*$\alpha\hat{g}$*-open and* $\hat{\omega}$*-open).The family of all* $\hat{\omega}$*-open sets of X(containing a point x* $\in$ *X) is denoted by* $\hat{\omega}O(X)(\hat{\omega}O(X,x))$*.By a multifunction* $F : X \to Y$*,we mean a point-to-set correspondence from X into Y and we always assume that F(x)* $\neq \phi$ *for all* $x \in X$*.For a multifunction* $F : X \to Y$*,we shall denote the upper and lower inverse of a set B of Y by* $F^+(B)$ *and* $F^-(B)$ *respectively,that is,* $F^+(B) = \{x \in X : F(x) \subset B\}$ *and* $F^-(B) = \{x \in X : F(x) \cap B \neq \phi\}$*.For each* $A \subset X$, $F(A) = \bigcup_{x \in A} F(x)$*.Also F is said to be a surjection if F(X)=Y.For a multifunction* $F : X \to Y$*,the graph multifunction* $G_F : X \to X \times Y$ *is defined as follows:* $G_F(x) = \{x\} \times F(x)$ *for every* $x \in X$*.If* $F_1 : X \to Y$ *and* $F_2 : Y \to Z$ *are multifunctions, then the composite multifunction* $F_2 \circ F_1 : X \to Z$ *is defined by* $(F_2 \circ F_1)(x) = F_2(F_1(x))$*for each* $x \in X$.

Lemma 2.3 *[7] For a multifunction* $F : X \to Y$
(i) $G_F^+(A \times B) = A \cap F^+(B)$ *(ii)* $G_F^-(A \times B) = A \cap F^-(B)$ *for any subsets* $A \subset X$ *and* $B \subset Y$.

Definition 2.4 *A multifunction $F : X \to Y$ is called*

(i)upper semi-continuous [8](resp.,lower semi-continuous) if $F^+(V)$(resp.,$F^-(V)$) is open in X for every open set V of Y.

(ii)upper strongly θ-continuous at $x \in X$ [4] if for each open set V with $F(x) \subset V$,there exists a θ-open set U containing x such that $F(U) \subset V$.

(iii)lower strongly θ-continuous [4]at $x \in X$ if for each open set V with $F(x) \cap V \neq \phi$,there exists a θ-open set U containing x such that $F(z) \cap V \neq \phi$ for each $z \in U$.

(iv)upper super continuous [1](resp.,lower super-continuous) if $F^+(V)$(resp.,$F^-(V)$) is δ-open in X for every open set V of Y.

(v)upper na-continuous [9] (resp.,lower na-continuous) if $F^+(V)$(resp.,$F^-(V)$) is δ-open in X for every α-open set V of Y.

3 Characterizations

In this section,we introduce upper(lower) $\hat{\omega}$-continuous multifunctions and we obtain many characterizations and basic properties of these multifunctions.

Definition 3.1 *A multifunction $F : X \to Y$ is called*

(i) upper $\hat{\omega}$-continuous at $x \in X$ if for each open set V of Y with $F(x) \subset V$,there exists $U \in \hat{\omega}O(X, x)$ such that $F(U) \subset V$.

(ii)lower $\hat{\omega}$-continuous at $x \in X$ if for each open set V of Y with $F(x) \cap V \neq \phi$,there exists $U \in \hat{\omega}O(X, x)$ such that $F(z) \cap V \neq \phi$ for each $z \in U$.

(iii)upper(lower) $\hat{\omega}$-continuous if F has this property at each point of X.

Example 3.2 *Let X=\{0,1,2\},$\tau = \{\phi, \{0\}, \{1\}, \{0, 1\}, X\}$ and Y=\{a,b,c,d\}, $\sigma = \{\phi, \{a, d\}, \{a, c, d\}, \{a, b, d\}, Y\}$.Define $F : X \to Y$ by F(0)=\{d\},F(1)=\{a\} and $F(2) = \{b, d\}$.Then F is upper $\hat{\omega}$-continuous.*

Example 3.3 *Let X=\{a,b,c\}=Y,$\tau = \{\phi, \{a\}, \{b, c\}, X\}$and $\sigma = \{\phi, \{a\}, \{a, b\}, Y\}$.Define $F : X \to Y$ by F(a)=\{a\},F(b)=\{c\} and F(c)=\{b\}.Then F is lower $\hat{\omega}$-continuous.*

Theorem 3.4 *The following are equivalent for a multifunction $F : X \to Y$:*

(i)F is upper $\hat{\omega}$-continuous.

(ii)for each $x \in X$ and for each open set V of Y such that $x \in F^+(V)$,there exists a $\hat{\omega}$-open set U of X containing x such that $U \subset F^+(V)$.

(iii)for each $x \in X$ and for each closed set K of Y such that $x \in F^+(Y-K)$, there exists a $\hat{\omega}$-closed set H of X such that $x \in X-H$ and $F^-(K) \subset H$.

(iv)for each open set V of Y,$F^+(V)$ is $\hat{\omega}$-open.

(v)for each closed set K of Y,$F^-(K)$ is $\hat{\omega}$-closed.

(vi)for each $x \in X$ and for each neighbourhood V of F(x),$F^+(V)$ is a $\hat{\omega}$-neighbourhood of x.

(vii)for each $x \in X$ and for each neighbourhood V of F(x),there exists a $\hat{\omega}$-neighbourhood of U of x such that $F(U) \subset V$.

Proof :

(i)⇔(ii)Obvious

(ii)⇒(iii)Let $x \in X$ and K be a closed subset of Y such that $x \in F^+(Y-K)$.By(ii) there exists a $U \in \hat{\omega}O(X,x)$ such that $U \subset F^+(Y-K)$.Let H=X-U.Then H is a $\hat{\omega}$-closed set such that $x \in X-H$.Also $U \subset F^+(Y-K) = X - F^-(K)$ implies $F^-(K) \subset X-U = H$.

(iii)⇒(ii)Let $x \in X$ and V be an open subset of Y such that $x \in F^+(V)$.Let K=Y-V.Then K is a closed set of Y such that $x \in F^+(Y-K)$.By(iii)there exists a $\hat{\omega}$-closed set H such that $x \in X-H$ and $F^-(K) \subset H$.Let U=X-H.Then U∈ $\hat{\omega}$O(X,x).Also $F^-(K) \subset H$ implies $X-F^+(Y-K) \subset H$.Hence $X-H \subset F^+(Y-K)$ and so $U \subset F^+(V)$.

(ii)⇒(iv)Let V be any open subset of Y and let $x \in F^+(V)$.Then by (ii)there exists $U_x \in \hat{\omega}O(X,x)$ such that $U_x \subset F^+(V)$.Thus $F^+(V) = \bigcup_{x \in F^+(V)} U_x$. Since arbitrary union of $\hat{\omega}$-open sets is $\hat{\omega}$-open, $F^+(V) \in \hat{\omega}O(X)$.

(iv)⇒(ii)Let $x \in X$ and V be an open subset of Y such that $x \in F^+(V)$.By (iv) $F^+(V) \in \hat{\omega}O(X)$.Let U=$F^+(V)$.Then $U \in \hat{\omega}O(X,x)$ and $U \subset F^+(V)$.

(iv)⇒(v)Let U be a closed subset of Y.Then Y-U is an open subset of Y.By (iv) $F^+(Y-U) \in \hat{\omega}O(X)$.But $F^+(Y-U) = X-F^-(U)$.Hence $X-F^-(U) \in \hat{\omega}O(X)$ which implies $F^-(U) \in \hat{\omega}C(X)$.

(v)⇒(iv)Let V be an open subset of Y. Then Y-V is a closed subset of Y.By (iv)$F^-(Y-V) \in \hat{\omega}C(X)$.But $F^-(Y-V) = X-F^+(V)$.Hence $X-F^+(V) \in \hat{\omega}C(X)$ which implies $F^+(V) \in \hat{\omega}O(X)$.

(iv)⇒(vi)Let $x \in X$ and V be a neighbourhood of $F(x)$.Then there exists an open set U of Y such that $F(x) \subset U \subset V$ which implies $x \in F^+(U) \subset F^+(V)$.By (iv) $F^+(U) \in \hat{\omega}O(X)$.Thus $F^+(V)$ is a $\hat{\omega}$-neighbourhood of x.

(vi)⇒(vii)Let $x \in X$ and V be a neighbourhood of $F(x)$.By (vi)$F^+(V)$ is a $\hat{\omega}$-neighbourhood of x.Take U=$F^+(V)$.Then U is a $\hat{\omega}$-neighbourhood of x such that $F(U) \subset V$.

(vii)⇒(i)Let $x \in X$ and V be an open subset of Y such that $F(x) \subset V$.Then V is a neighbourhood of $F(x)$.By (vii) there exists a $\hat{\omega}$-neighbourhood U of x such that $F(U) \subset V$.Then there exists $G \in \hat{\omega}O(X)$such that $x \in G \subset U$ and so $F(G) \subset F(U) \subset V$.Hence F is upper $\hat{\omega}$-continuous.

Similarly,we can obtain the following characterizations for lower $\hat{\omega}$-continuous multifunctions.

Theorem 3.5 *The following are equivalent for a multifunction $F : X \rightarrow Y$:*

(i)F is lower $\hat{\omega}$-continuous.

(ii)for each $x \in X$ and for each open set V of Y such that $x \in F^-(V)$,there exists a $\hat{\omega}$-open set U of X containing x such that $U \subset F^-(V)$.

(iii)for each $x \in X$ and for each closed set K of Y such that $x \in F^-(Y - K)$,there exists a $\hat{\omega}$-closed set H of X such that $x \in X - H$ and $F^+(K) \subset H$.

(iv)for each open set V of Y, $F^-(V)$ is $\hat{\omega}$-open.

(v)for each closed set K of Y, $F^+(K)$ is $\hat{\omega}$-closed.

Remark 3.6 *From [1] we have the following implication :upper(lower) strongly θ-continuous ⇒upper(lower) super continuous,but not conversely.*

Remark 3.7 *From [9] we have the following implication:upper(lower) na-continuous ⇒upper(lower)super continuous ⇒upper(lower)semi-continuous, but not conversely.*

Theorem 3.8 *Every upper(lower) super multifunction is upper(lower) $\hat{\omega}$-continuous.*
Proof: *Since every δ-open set is $\hat{\omega}$-open[6],the proof follows.*

Theorem 3.9 *Every upper(lower)strongly θ-continuous multifunction is upper(lower) $\hat{\omega}$-continuous.*
Proof : *Since every θ-open set is $\hat{\omega}$-open[6],the proof follows.*

Theorem 3.10 *Every upper(lower)na-continuous multifunction is upper(lower) $\hat{\omega}$-continuous.*

Proof : *Let V be an open set in Y. Then V is an α-open set in Y. Since F is upper na-continuous, $F^+(V)$ is δ-open in X. Since every δ-open set is $\hat{\omega}$-open[6], the proof follows.*

Remark 3.11 *None of the above implications is reversible as shown in the following examples.*

Example 3.12 *Let $X=\{a,b,c\}=Y$, $\tau = \{\phi, \{a,b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b,c\}, Y\}$. Define a function $F : X \to Y$ by $F(a)=\{a\}$, $F(b)=\{c\}$ and $F(c) = \{a,b\}$. Then F is upper $\hat{\omega}$-continuous but not upper strongly θ-continuous.*

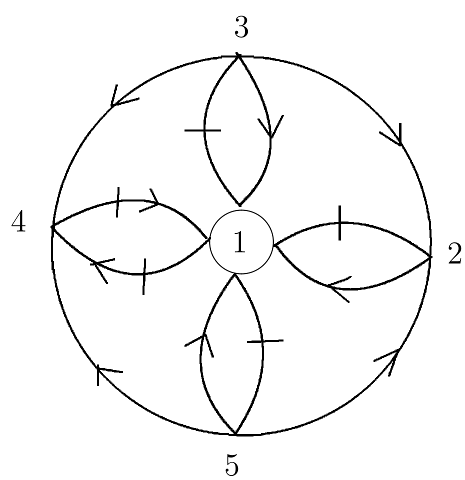
Example 3.13 *Let $X=\{a,b,c\}=Y$, $\tau = \{\phi, \{a,b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a,b\}, Y\}$. Define a function $F : X \to Y$ by $F(x)=\{x\}$ for every $x \in X$. Then F is upper $\hat{\omega}$-continuous but neither upper super continuous nor upper na-continuous.*

Remark 3.14 *Upper $\hat{\omega}$-continuous and upper semi-continuous are independent of each other as shown below.*

Example 3.15 *Let $X=\{a,b,c\}$, $Y=\{p,q,r,s\}$, $\tau = \{\phi, \{a,b\}, X\}$ and $\sigma = \{\phi, \{p,s\}, \{p,r,s\}, \{p,q,s\}, Y\}$. Define a function $F : X \to Y$ by $F(a)=\{s\}$, $F(b)=\{r,s\}$ and $F(c)=\{q,r\}$. Then F is upper $\hat{\omega}$-continuous but not upper semi-continuous.*

Example 3.16 *Let $X=\{a,b,c,d\}$, $Y=\{p,q,r,s\}$, $\tau = \{\phi, \{a\}, \{a,b\}, \{a,c,d\}, X\}$ and $\sigma = \{\phi, \{p,s\}, \{p,r,s\}, \{p,q,s\}, Y\}$. Define a function $F : X \to Y$ by $F(a)=\{p\}$, $F(b)=\{q,s\}$, $F(c)=\{q,r\}$ and $F(d)=\{q,r,s\}$. Then F is upper semi-continuous but not upper $\hat{\omega}$-continuous.*

Remark 3.17 *From the above results we have the following diagram where $A \to B$ represents A implies B but not conversely and $A \nrightarrow B$ represents A does not imply B.*



1.upper $\hat{\omega}$-continuous 2.upper super continuous 3.upper na-continuous 4.upper semi-continuous 5.upper strongly θ-continuous

Theorem 3.18 *Let X, Y and Z be topological spaces and let $F : X \to Y$ and $G : Y \to Z$ be multifunctions.If F is upper(lower) $\hat{\omega}$-continuous and G is upper(lower) semi-continuous,then $G \circ F : X \to Z$ is upper(lower) $\hat{\omega}$-continuous.*
Proof : *Let W be an open set in Z.Since G is upper(lower)semi-continuous, $G^+(W)(G^-(W))$ is an open set in Y.In view of upper(lower)$\hat{\omega}$-continuity of F, $F^+(G^+(W))(F^-(G^-(W))$ is an $\hat{\omega}$-open set in X and so the multifunction $G \circ F$ is upper(lower) $\hat{\omega}$-continuous.*

Remark 3.19 *From theorem 3.8 ,theorem 3.9 and theorem 3.10, the above theorem 3.18 is true even if F is upper(lower) super continuous or upper(lower) strongly θ-continuous or upper(lower) na-continuous.*

Theorem 3.20 *Let $F : X \to Y$ be a multifunction.If the graph multifunction G_F is upper $\hat{\omega}$-continuous then F is upper $\hat{\omega}$-continuous where $G_F : X \to X \times Y$ is defined by $G_F(x) = \{x\} \times$ F(x).*
Proof: *Let $x \in X$ and V be any open set of Y containing F(x).Then $X \times V$ is open in $X \times Y$ and $G_F(x) \subset X \times V$.Since G_F is upper $\hat{\omega}$-continuous,there exists $U \in \hat{\omega}O(X, x)$ such that $G_F(U) \subset X \times V$.Hence $U \subset G_F^+(X \times V)$.By lemma 2.3,$G_F^+(X \times V) = F^+(V)$ and so $U \subset F^+(V)$.Thus F is upper $\hat{\omega}$-continuous.*

Theorem 3.21 *Let $F : X \to Y$ be a multifunction.If the graph multifunction G_F is lower $\hat{\omega}$-continuous then F is lower $\hat{\omega}$-continuous.*
Proof : *Let $x \in X$ and V be any open set of Y such that $x \in F^-(V)$.Then $X \times V$ is open in $X \times Y$.Also $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \phi$.Since G_F is lower $\hat{\omega}$-continuous,there exists $U \in \hat{\omega}O(X, x)$ such that $U \subset G_F^-(X \times V)$.By lemma 2.3,$G_F^-(X \times V) = F^-(V)$ and so $U \subset F^-(V)$.Thus F is lower $\hat{\omega}$-continuous.*

Theorem 3.22 *Suppose that (X, τ) and (X_α, τ_α) are topological spaces where $\alpha \in \Lambda$.Let $F : X \to \prod_{\alpha \in \Lambda} X_\alpha$ be a multifunction from X to the product space $\prod_{\alpha \in \Lambda} X_\alpha$ and let $P_\alpha : \prod_{\alpha \in \Lambda} X_\alpha \to X_\alpha$ be the projection multifunction for each $\alpha \in \Lambda$ which is defined by $P_\alpha((x_\alpha)) = \{x_\alpha\}$.If F is upper (lower) $\hat{\omega}$-continuous, then $P_\alpha \circ F$ is upper(lower) $\hat{\omega}$-continuous for each $\alpha \in \Lambda$.*
Proof: *Let V_α be an open set in (X_α, τ_α) where $\alpha \in \Lambda$.Then $(P_\alpha \circ F)^+(V_\alpha) = F^+(P_\alpha^+(V_\alpha)) = F^+(V_\alpha \times \prod_{\beta \neq \alpha} X_\beta)$(resp.$(P_\alpha \circ F)^-(V_\alpha) = F^-(P_\alpha^-(V_\alpha)) = F^-(V_\alpha \times \prod_{\beta \neq \alpha} X_\beta)$.Since F is upper (lower) $\hat{\omega}$-continuous and $V_\alpha \times \prod_{\beta \neq \alpha} X_\beta)$ is an open set, from theorem 3.4 and 3.5 we get $F^+(V_\alpha \times \prod_{\beta \neq \alpha} X_\beta)$(resp.$F^-(V_\alpha \times \prod_{\beta \neq \alpha} X_\beta))$ is $\hat{\omega}$-open in X.Hence by theorem 3.4 and 3.5, P_α is upper(lower) $\hat{\omega}$-continuous for each $\alpha \in \Lambda$.*

Theorem 3.23 *Suppose that (X,τ),(Y,σ) and (Z,η) be topological spaces and let $F_1 : X \to Y$ and $F_2 : X \to Z$ be multifunctions.Let $F_1 \times F_2 : X \to Y \times Z$ be a multifunction defined by $(F_1 \times F_2)(x) = F_1(x) \times F_2(x)$ for each $x \in X$. If $F_1 \times F_2$ is an upper (lower) $\hat{\omega}$-continuous multifunction, then $F_1$and F_2 are upper (lower) $\hat{\omega}$-continuous multifunctions.*
Proof : *Let $x \in X$ and let $V \subset Y$ and $W \subset Z$ be open sets such that $x \in F_1^+(V)$ and $x \in F_2^+(W)$.Then $F_1(x) \subset V$ and $F_2(x) \subset W$.Hence $(F_1 \times F_2)(x) = F_1(x) \times F_2(x) \subset V \times W$and so $x \in (F_1 \times F_2)^+(V \times W)$.Since $F_1 \times F_2$ is an upper $\hat{\omega}$-continuous multifunction,there exists a $\hat{\omega}$-open set U containing x such that $U \subset (F_1 \times F_2)^+(V \times W)$ which implies $U \subset F_1^+(V)$ and $U \subset F_2^+(W)$.Hence F_1 and F_2 are upper (lower) $\hat{\omega}$-continuous multifunctions.*
The proof for lower $\hat{\omega}$-continuous multifunctions is similar.

4 Applications

Definition 4.1 *A space X is $\hat{\omega}$-compact[5] if every $\hat{\omega}$-open cover of X has a finite subcover.*

Theorem 4.2 *Let $F : X \to Y$ be an upper $\hat{\omega}$-continuous surjective multifunction such that F(x) is compact relative to Y for each $x \in X$.If X is a $\hat{\omega}$-compact space,then Y is compact.*
Proof : *Let $\{V_\alpha : \alpha \in \Lambda\}$ be an open cover of Y.Since F(x) is compact relative to Y for each $x \in X$,there exists a finite subset $\Lambda(x)$of Λ such that $F(x)$ $\subset\{V_\alpha : \alpha \in \Lambda(x)\}$.Let $V(x) = \bigcup\{V_\alpha : \alpha \in \Lambda(x)\}$.Then V(x) is an open subset of Y containing F(x).Since F is upper $\hat{\omega}$-continuous, there exists a $\hat{\omega}$-open set U(x) of X such that $F(U(x)) \subset V(x)$.Then the family $\{U(x) : x \in X\}$ is a $\hat{\omega}$-open cover of X and since X is $\hat{\omega}$-compact,there exists $x_1, x_2, ...x_n$ such that $X = \bigcup_{i=1}^n\{U(x_i) : i = 1, 2, ...n\}$.Hence $Y = F(X) = F(\bigcup_{i=1}^n U(x_i)\ ) = \bigcup_{i=1}^n F(U(x_i)) \subset \bigcup_{i=1}^n V(x_i) = \bigcup_{i=1}^n \bigcup_{\alpha\in\Lambda(x_i)} V_\alpha$ which shows that Y is compact.*

Definition 4.3 *A multifunction $F : X \to Y$ is punctually connected if for each $x \in X$,F(x) is connected.*

Definition 4.4 *A space X is called $\hat{\omega}$-connected[5] provided X is not the union of two disjoint nonempty $\hat{\omega}$-open sets.*

Theorem 4.5 *Let $F : X \to Y$ be an upper $\hat{\omega}$-continuous,surjective multifunction.If X is $\hat{\omega}$-connected and F is punctually connected. then Y is connected.*
Proof : *Suppose Y is not connected.Then there exists disjoint open sets U and V such that $Y = U \cup V$.Since F(x) is connected for each $x \in X$,either*

$F(x) \subset U$ or $F(x) \subset V$.This implies either $x \in F^{+}(U)$ or $x \in F^{+}(V)$ and hence $F^{+}(U) \cup F^{+}(V) = X$.Since $U \neq \phi$,we may choose $u \in U$.Since F is surjective,there exists $x \in X$ such that $u \in F(x)$ and so $F(x) \subset U$.Hence $x \in F^{+}(U)$ which implies $F^{+}(U) \neq \phi$.In the same way $V \neq \phi$ implies $F^{+}(V) \neq \phi$.Also $F^{+}(U) \cap F^{+}(V) = \phi$.Since F is upper $\hat{\omega}$-continuous,$F^{+}(U)$and $F^{+}(V)$ are $\hat{\omega}$-open sets in X, a contradiction to X is $\hat{\omega}$-connected.

Recall that a multifunction $F : X \rightarrow Y$ is said to be punctually closed if for each $x \in X$, F(x) is closed.

Definition 4.6 *A topological space X is said to be $\hat{\omega} - T_2$[5] if for each pair of distinct points x and y of X,there exist disjoint $\hat{\omega}$-open sets U and V of X containing x and y respectively.*

Theorem 4.7 *Let $F : X \rightarrow Y$ be an upper $\hat{\omega}$-continuous multifunction and punctually closed from a topological space X into a normal space Y such that $F(x) \cap F(y) = \phi$ for each pair of distinct points x and y of X.Then X is a $\hat{\omega} - T_2$ space.*

Proof : *Let x and y be distinct points of X.Then $F(x) \cap F(y) = \phi$.Since Y is normal and F is punctually closed,there exist disjoint open sets U and V containing F(x) and F(y) respectively.Now $F(x) \subset U$ and $F(y) \subset V$ respectively implies $x \in F^{+}(U)$ and $y \in F^{+}(V)$.Since F is upper $\hat{\omega}$-continuous,by theorem 3.4 $F^{+}(U)$ and $F^{+}(V)$ are disjoint $\hat{\omega}$-open sets of X containing x and y respectively.Consequently X is $\hat{\omega} - T_2$.*

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