International Mathematical Forum, Vol. 7, 2012, no. 52, 2603 - 2611

Some Properties of Composition Operators on Weighted Hardy Spaces

L. Karimi

Islamic Azad University, Shahrekord Branch, Shahrekord, Iran
lkarimi@iaushk.ac.ir

Abstract

Let φ be an analytic map of unit disk $\mathbb{D}$ into itself, consider the composition operator C_φ defined by $C_\varphi(f) = f \circ \varphi$ whenever f is analytic on $\mathbb{D}$. In this paper, we discuss necessary and sufficient conditions under which a composition operator on a large class of weighted Hardy spaces is a compact.

1 Introduction and Preliminaries

An analytic self-map $\varphi : \mathbb{D} \to \mathbb{D}$ of the open unit disk $\mathbb{D}$ in the complex plane induces the composition operator C_φ on $H(\mathbb{D})$, the space of holomorphic functions on $\mathbb{D}$, defined by $C_\varphi(f) = \varphi \circ f$. A basic goal in the study of composition operators is to relate function theoretic properties of φ to operator theoretic properties of C_φ. Here we review some results that characterize when φ induces a compact composition operator between various classical Banach spaces of holomorphic functions on $\mathbb{D}$.
For $p > 0$, the Hardy space $H^p(D)$ is the space of functions f that are analytic on $\mathbb{D}$ and satisfy

$$\lim_{r\to 1^-} \int_0^{2\pi} |f_r(e^{i\theta})|^p \frac{d\theta}{2\pi} < \infty,$$

Denote the p^{th} root of this limit by $\|f\|_p$.The Hard space $H^\infty(D)$ is the set of analytic functions that are bounded in D, with supremum norm $\|f\|_\infty$ and the Bergman space $A^p(D)$ consists of those analytic functions such that

$$\int_D |f(z)|^p \frac{dA(z)}{\pi} < \infty$$

Here dA denotes area measure on $\mathbb{D}$ by, normalized so that $A(\mathbb{D}) = 1$, similarly $\|f\|_p$ is the p^{th} root of this integral.

In the following the sufficient condition for compactness of composition operator C_φ in standard Hardy space H^2 is given.

The weighted Hardy space $H^2(\beta)$ is a Hilbert space whose vectors are functions analytic on the unit disk and monomials $1, z, z^2, ...$ form a complete orthogonal set of non-zero vectors. If $\|z^j\| = \beta(j)$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then $f \in H^2(\beta)$ if and only if

$$\|f\|_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta(n)^2 < \infty$$

The weighted Bergman space defined for $\alpha > -1$ by

$$A_\alpha^p(D) = \{f \ analytic \ in \ D \ : \quad \int_D |f(z)|^p (1-|z|^2)^\alpha \frac{dA(z)}{\pi} < \infty\}$$

The linear functional K_x with $K_x(f) = f(x)$ is called reproducing kernel. The generating function for the weighted Hardy space $H^2(\beta)$ is the function $k(z) = \sum_{j=0}^{\infty} \frac{z^j}{\beta(j)^2}$. This function is analytic on D. Moreover, if $w \in \mathbb{D}$ then $K_w(z) = k(\bar{w}z)$ and $\|K_w\|^2 = k(|w|^2)$, (see [1]).
Recently, there has been a great interest in studying operator theoretic properties of composition and weighted composition operators, see, for example, monographs [1] and [4], papers [2], [3], [7] and [8] as well as the reference therein.

Theorem 1. *In a weighted Hardy space for which the series $\sum \beta(n)^{-2}$ diverges. the normalized reproducing kernels*

$$\frac{K_\omega}{\|K_\omega\|}$$

tends to 0 weakly as $|\omega|$ tends to 1.

The Reproducing kernels for elements near of the $\partial\mathbb{D}$ for weighted Hardy spaces has important consequences. For example we show that the generating function is smooth on the closed unit disk.

Theorem 2. *If $H^2(\beta)$ is a weighted Hardy space for which the generating function k is continuous on the closed disk then all functions in $H^2(\beta)$ can be extended continuously to the closed disk.*

Proof. Let $f \in H^2(\beta), w_0 \in \partial\mathbb{D}$ we show that $\lim_{w \to w_0}$is exist. To show this let $z \in \bar{\mathbb{D}}$

$$K_{w_0}(z) = \lim_{w \to w_0} = \lim_{w \to w_0} k(\bar{w}z) = k(\bar{w_0}z) < \infty$$

and hence $\|K_w - K_{w_0}\| \to 0 \ as \ w \to w_0$. On the other hand $\|K_w\|^2 = k(|w|^2)$ thus $\|K_{w_0}\|^2 = k(|w_0|^2)$ and then

$$\begin{aligned}\|K_w - K_{w_0}\|^2 &= \|K_w\|^2 - 2Re\langle K_{w_0}, K_w\rangle + \|K_{w_0}\|^2 \\ &= \|K_w\|^2 - 2ReK_w(w_0) + \|K_{w_0}\|^2 \to 0 \ as\ w \to w_0\end{aligned}$$

thus

$$|f(w) - \langle f, K_{w_0}\rangle| = |\langle f, K_w - K_{w_0}\rangle| \leq \|f\|\|K_w - K_{w_0}\| \to 0$$

So $\lim_{w\to w_0} f(w) = \langle f, K_{w_0}\rangle = f(w_0)$ □

In follow we give an example of a Hilbert space of analytic functions on $\mathbb{D}$ consisting continuous functions on $\bar{\mathbb{D}}$ but we can find at least one function that is not differentiable.

Example 1. *Consider the Hardy space $H^2(\beta)$ containing all analytic functions f on the unit disk for which*

$$\|f\|^2 = \sum_{j=0}^{\infty} \beta(j)^2 |a_j|^2 < \infty$$

where $\beta(0) = 1$, $\beta(j) = j$. It is clearly $k(z)$ is continuous on $\bar{\mathbb{D}}$ because $k(z) = \sum_{j=1}^{\infty} \frac{z^j}{\beta(j)^2} = \sum_{j=1}^{\infty} \frac{z^j}{j^2}$ by M-test weierestrass this series is uniformly convergence on $\bar{\mathbb{D}}$ thus by last Theorem each function in $H^2(\beta)$ extends to continuous function on $\bar{\mathbb{D}}$. The radius convergence of f is ≥ 1 thus $f'(z) = \sum_{j=1}^{\infty} \frac{z^{j-1}}{j}$ since $\sum \frac{1}{j}$ is divergence, f is not differentiable at 1.

2 Necessary and sufficient conditions for Compactness

A linear operator on a Banach space is compact if the image of the unit ball under the operator has compact closure. This definition has a good reformation. Indeed If $\mathcal{X}$ is the weighted Hardy space$H^2(\beta)$, The Hardy space $H^p(D)$ or $A^p_\alpha(D)$ for for $0 < p \leq \infty$ and $\alpha > -1$. Then C_φ is compact on $\mathcal{X}$ if and only if whenever $\{f_n\}$ is bounded in $\mathcal{X}$ and $f_n \to 0$ uniformly on compact subsets of D then $C_\varphi f_n \to 0$ in $\mathcal{X}$.

Proposition 1. *If $\|\varphi\|_\infty < 1$ then C_φ is compact on H^2.*

Proof. Suppose $\{f_n\}$ is a bounded sequence in H^2 that converges to zero uniformly on compact subset of D. By Theorem (2.1.12) it is enough to show that $\|f_n \circ \varphi\| \longrightarrow 0$. But even more is true: since $\varphi(D)$ is a relatively compact subset of D and $f_n \longrightarrow 0$ uniformly on $\varphi(D)$, hence

$$\begin{aligned}\|f_n \circ \varphi\| \leq \|f_n \circ \varphi\|_\infty &= \sup_{z\in D} |(f_n \circ \varphi)(z)| \\ &\leq \sup_{w\in\varphi(D)} |f_n(w)| \longrightarrow 0.\end{aligned}$$

as desired. □

Corollary 1. *Consider $H^p(D)$ or $A^p_\alpha(D)$ for $0 < p \leq \infty$ and $\alpha > -1$. If $\overline{\varphi(D)} \subset D$ then C_φ is compact.*

Proof. Suppose that $\{f_n\}$ is a bounded sequence and $f_n \to 0$ uniformly on compact subsets of D since $\overline{\varphi(D)} \subseteq D$ is compact thus $f_n \circ \varphi \to 0$ uniformly and hence C_φ is compact. □

Corollary 2. *C_φ is compact on $H^\infty(D)$ if and only if $\overline{\varphi(D)} \subset D$.*

Proof. Suppose that C_φ is compact on $H^\infty(\mathbb{D})$, and $\overline{\varphi(D)} \nsubseteq D$, so there exist $w_0 \in \overline{\varphi(D)}$ such that $w_0 \in w_0$ so $|w_0| = 1$ thus there exist a sequence $\{z_n\}$ in $\mathbb{D}$ such that $\varphi(z_n) \to w_0$. Thus $\|\varphi\|_\infty = 1$. On the other hand, if $f_n(z) = z^n$, $f_n \to 0$, uniformly on compact subsets of $\mathbb{D}$. But C_φ is compact, so by proposition? $f_n \circ \varphi \to 0$ in $H^\infty(\mathbb{D})$, that means $\varphi^n \to 0$ in $H^\infty(\mathbb{D})$ or $\|\varphi^n\|_\infty \to 0$. It is a contradiction since $\|\varphi\|_\infty = 1$, and consequently for every n we have $\|\varphi^n\|_\infty = 1$ so we must have $\overline{\varphi(D)} \subset D$.
Conversely suppose that $\overline{\varphi(D)} \subset D$, and $\{f_n\}$ is a bounded sequence in $H^\infty(\mathbb{D})$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Since $\overline{\varphi(\mathbb{D})}$ is compact thus $f_n \to 0$ uniformly on $\overline{\varphi(\mathbb{D})}$. Since $\{|f_n(w)| : w \in \varphi(\mathbb{D})\} \subseteq \{|f_n(w)| : w \in \overline{\varphi(\mathbb{D})}\}$ then $\|f_n \circ \varphi\|_\infty \to 0$ *as* $n \to \infty$. □

An equivalence necessary condition for compactness of composition operator C_φ on Hardy space $H^p(D)$ or $A^p_\alpha(D)$ with respect to Jullia-caratheodory Theorem so makes the relationship between the angular derivative $\varphi'(xi)$ and the evalue $d(\xi)$ is following.

Corollary 3. *If C_φ is compact on $H^p(D)$ or $A^p_\alpha(D)$ then φ has no finite angular derivative at any point of ∂D.*

Theorem 3 (P.Ahem and D.Clark). *The singular inner function*

$$\varphi(z) = exp \int_{\partial\mathbb{D}} \frac{z+\xi}{z-\xi} d\mu(\xi)$$

for μ singular to lebesgue measure has finite angular derivative at η in $\partial\mathbb{D}$ if and only if

$$\int_{\partial\mathbb{D}} \frac{d\mu(\xi)}{|\xi - \eta|^2} < \infty.$$

Corollary 4. *There exist inner functions that induce compact composition operators on $A^p_\alpha(\mathbb{D})$, for $\alpha > -1$ and $0 < p < \infty$.*

Proof. Choose positive numbers $(\mu_n)_n$ such that $\sum_n \mu_n < \infty$, but $\sum_n \sqrt{\mu_n} = \infty$. (for example $\mu_n = \frac{1}{n^2}$). Let $(I_n)_n$ be a sequence of consecutive arcs on $\partial\mathbb{D}$ with $l(T_n) = \sqrt{\mu_n}$, and let ξ_n be the center of T_n. Let $\mu = \sum_n \mu_n \delta_n$, where δ_n is the unit mass at ξ_n.
Now we show that for every $w \in \partial\mathbb{D}$,w belongs to infinitely many intervals I_n, hence $|w - \xi_n| < \sqrt{\mu_n}$for infinitely many n, so the series; $\sum_n \frac{\mu_n}{|\xi_n - w|^2}$ diverges, since infintely many terms are > 1. But

$$\int_{\partial\mathbb{D}} \frac{d\mu(\xi)}{|\xi - w|^2} = \sum_n \frac{\mu_n}{|\xi_n - w|^2} = \infty.$$

Hence by last Theorem φ has no finite angular derivative at any point of ∂D and then C_φ is compact on $A^p_\alpha(\mathbb{D})$. □

In the following we show that the converse of corollary 3. is not necessary valid.Indeed we give an example on $H^p(D)$ such that map φ maps $\mathbb{D}$ into itself, has no finite angular derivative and C_φ is not compact on $H^p(D)$.

Example 2. *Let ψ_0 be an inner function as in Theorem 3. Let $a = \psi_0(0)$, consider*

$$\psi(z) = \frac{a - \psi_0(z)}{1 - \bar{a}\psi_0(z)} \quad \forall z \in \mathbb{D}.$$

It is clearly $\psi(0) = 0$. Define $\varphi(z) = \frac{1+\psi(z)}{2}$ for all $z \in \mathbb{D}$; clearly $|\varphi^*(\xi)| < 1$ almost every where.
We have $\varphi = \chi o \psi$, where $\chi(z) = \frac{1+z}{2}$, then $C_\varphi = C_\psi o C_\chi$ since χ has finite angular derivative at every point of $\partial\mathbb{D}$ so C_χ is no compact, so it takes the unit ball of H^p into a set A whose closure is not compact, but ψ_0 is an inner, which fixes the origin, so C_φ is an isometry of H^p, so $C_\psi(A)$ does not have compact closure either. But $C_\psi(A) = C_\varphi(ballH^p)$ so C_φ is not compact operator

Now we put many conditions on weighted sequence $\beta(n)$ such that weighted Hardy space $H^2(\beta)$ become "small" or "large". Although the terminology small weighted Hardy space is vary somewhat from theorem to theorem, but one of this conditions is $\sum_{n=0}^{\infty} \frac{1}{\beta(n)^2} < \infty$, on each other if weight $\beta(n)$ on $H^2(\beta)$ tends to 0 sufficiently rapidly so that $\lim_{n\to\infty} n^a \beta(n) = 0$, for all $a > 0$ $H^2(\beta)$ is called large weighted Hardy space.
Kreite and MacCluer in [3] proved in such large weighted Hardy space if the symbol function φ has angular derivative in D at some point in the circle then composition operator C_φ does not map $H^(\beta)$ into itself. In the following a dual result of Kreite and MacCluer Theorem is obtained.

Theorem 4. *Suppose $\varphi : D \longrightarrow D$ is analytic with $|\varphi'(\xi)| > 1$ for some $\xi \in \partial D$ satisfying $|\varphi(\xi)| = 1$. Then C_φ is not compact on $H^2(\beta)$ when ever $\{\beta(n)\}$ satisfies*

$$\sum_{n=0}^{\infty} \frac{1}{\beta(n)^2} < \infty.$$

Proof. Suppose $\varphi(\xi) = \eta \in \partial D$. Let $\psi(z) = \xi\bar{\eta}\varphi(z)$ then we have $\psi(\xi) = \xi\bar{\eta}\varphi(\xi) = \xi\bar{\eta}\eta = \xi$ and $\psi'(\xi) = |\psi'(\xi)| = |\varphi'(\xi)| > 1$. Since ψ is analytic self-map of D if ψ has no fixed point in D then by Wolff's Theorem there is a unique point on ∂D such that φ has angular derivative less than or equal one. Therefore this point can not be ξ. Call this point a. We have

$$C_\psi^*(K_\xi) = K_{\psi(\xi)} = K_\xi \quad , \quad C_\psi^*(K_a) = K_{\psi(a)} = K_a,$$

where K_w denotes the kernel function for evaluation at $w \in \bar{D}$. Thus if C_φ, and hence C_ψ, is compact on $H^2(\beta)$, then $\dim\ker(C_\psi - 1) = \dim\ker(C_\psi^* - 1) \geq 2$. But if $f \in \ker(C_\psi - 1)$ then $C_\psi f = f$ and therefore $f \circ \psi_n = f$, where ψ_n is the nth iterate of ψ.Now by Denjoy-Wolff theorem and continuity of f on $\bar{D}$ implies that f is constant.
If ψ has a fixed point in D, since ψ is not an elliptic automorphism of D and by proportion (2.4.1) implies that f is constant and its contradiction with $\dim\ker(C_\varphi - 1) \geq 2$. Thus C_φ can not be compact on $H^2(\beta)$. □ The next result

applies to small spaces defined slightly more restrictively by requiring that functions in the space have derivative which extends continuously to $\bar{D}$. □

Theorem 5. *Suppose $\varphi : D \longrightarrow D$ is analytic with $|\varphi'(\xi)| = 1$ for some $\xi \in \partial D$ with $|\varphi(\xi)| = 1$. If*

$$\sum_{n=0}^{\infty} \frac{n^2}{\beta(n)^2} < \infty,$$

then C_φ is not compact on $H^2(\beta)$.

Proof. We normalize by choosing $e^{i\theta}$ such that $\psi = e^{i\theta}\varphi$ fixes ξ, therefore $\psi(\xi) = \xi$. By theorem (2.4.3) we have

$$\psi'(\xi) = |\psi'(\xi)| = |e^{i\theta}\varphi'(\xi)| = |\varphi'(\xi)| = 1.$$

Thus $\psi'(\xi) = 1$. As before $C_\psi^*(K_\xi) = K_\xi$, where K_ξ is the kernel function for evaluation at ξ.

If $K_\xi^{(1)}$ is the kernel function for the evaluation of the first derivative at ξ, then we have

$$K_\xi(z) = \sum_{n=0}^{\infty} \frac{1}{\beta(n)^2} (\bar{\xi} z)^n,$$

and

$$K_\xi^{(1)} = \frac{d}{d\bar{\xi}} K_\xi,$$

therefore

$$K_\xi^{(1)}(z) = \sum_{n=1}^{\infty} \frac{n}{\beta(n)^2} \bar{\xi}^{n-1} z^n.$$

$K_\xi^{(1)}(z)$ is a function in $H^2(\beta)$ and for $f \in H^2(\beta)$ we have

$$\begin{aligned} < f, K_\xi^{(1)}(z) >_\beta &= \sum_{n=0}^{\infty} a_n \frac{n}{\beta(n)^2} \xi^{n-1} \beta(n)^2 \\ &= \sum_{n=0}^{\infty} n a_n \xi^{n-1} \\ &= f'(\xi). \end{aligned}$$

Thus

$$\begin{aligned} < f, C_\psi^* K_\xi^{(1)} >_\beta &= < C_\psi f, K_\xi^{(1)} >_\beta \\ &= < f \circ \psi, K_\xi^{(1)} >_\beta \\ &= (f \circ \psi)'(\xi) \\ &= \psi'(\xi).f'(\psi(\xi)) \\ &= \psi'(\xi) < f, K_{\psi(\xi)}^{(1)} >_\beta \\ &= < f, \overline{\psi'(\xi)} K_{\psi(\xi)}^{(1)} >_\beta \\ &= < f, K_\xi^{(1)} >_\beta . \end{aligned}$$

Therefore

$$C_\psi^* K_\xi^{(1)} = K_\xi^{(1)}$$

But we know that $C_\psi^* K_\xi = K_\xi$, thus 1 is an eigenvalue of C_ψ^* with multiplicity at least 2. As in the proof of theorem (2.4.3) this show that C_ψ (and hence C_φ) cannot be compact, since $\dim \ker(C_\psi - 1) = 1$. □ The next result, due to

J. Shapiro [17], provides a stepping stone for obtaining a necessary condition, in term of the angular derivative, for C_φ to be bounded on $H^2(\beta)$. □

In contrast to the function spaces considered in the last chapter, here we consider weighted Bergman or Hardy spaces that include functions which grow much more rapidly than functions in $H^2(D)$ or the standard weighted Bergman spaces $A^2_\alpha(D)$. We will confine our attention to function spaces on the disk, and concentrate on spaces $A^2_G(D)$ defined for weights G positive, continuous, and non-increasing on $(0,1)$ with $\int_0^1 G(r) < \infty$ so that $G(|z|)dA(z)$ is a positive, circularly symmetric finite measure on D. Set

$$A_G^2(D) \equiv \{f analytic: \int_D |f(z)|^2 G(|z|) \frac{dA(z)}{\pi}\}$$

with

$$||f||_G^2 = \int_D |f(z)|^2 G(|z|) \frac{dA(z)}{\pi}$$

The space A_G^2 is equivalent to the weighted Hardy space $H^2(\beta)$ where,$\beta(n)^2 = p_n/c$ where $\{p_n\}$are the moments

$$p_n = \int_0^1 r^{2n+1} G(r) dr$$

and $c = 2\int_0^1 G(r) r dr$,chosen so that $\beta(0) = 1$.

Theorem 6. *(Kriete and MacCluer) Let G be a regular fast weight. Then the following statement are equivalent.*
(i) C_φ *is compact on* A_G^2
(ii) $\lim_{|z|\to 1} G(|z|)/G(|\varphi(z)|) = 0$
(iii) $|\varphi'(\xi)| > 1$ *for all* ξ *in* ∂D
(iv) $\beta(\varphi) > 1$

Corollary 5. *If* $\varphi(0) = 0$ *and* φ *is not a rotation of* D*, then* C_φ *is compact on* $A_G^2(D)$*, where* G *is fast and regular.*

Proof. By Theorem 6. it is enough to show that $|\varphi'(\xi)| > 1$ for all ξ in ∂D. suppose ξ is in the unit circle and $|\varphi'(\xi)| \leq 1$ thus $d(\xi)$ is finite and then by julia's Lemma we conclude that $d(\xi) = 1$ that means equality holds for $0 \in D$ and then φ is an automorphism of the disk.Thus

$$\varphi(z) = \lambda \frac{a - z}{1 - \bar{a}z} \qquad |\lambda| = 1, \quad |a| < 1.$$

But $0 = \varphi(0) = \lambda a$ thus $a = 0$.This is a contradiction with φ is not a rotation. □

References

[1] C. C. Cowen and B. D. MacCluer, Composition operators on spaces of analytic functions, CRC Press, Boca Raton, 1995.

[2] K.Hedayatian and L. Karimi, On convexity of composition and multiplication operators on weighted Hardy spaces, Preprint.

[3] T. I. Kriete and B. D. MacCluer, Composition operators on large weighted Bergman spaces, Indiana Univ. Math. J. **41** (1992), 755-788.

[4] J. H. Shapiro, Composition operators and classical function theory, Springer-Verlag, New York, 1993.

[5] J. H. Shapiro, Compact composition operators on spaces of boundary-regular holomorphic functions, Proc. Amer. Math. Soc. **100** (1987), 49-57.

[6] J. H. Shapiro and C. Sundberg, Compact composition operators on L^1, Proc. Amer. Math. Soc. **108**(1990), 443-449.

[7] A. L. Shields, "Weighted shift operators and analytic function theory" in topics in operator theory, Math. Surveys, vol. **13**, Amer. Math. Soc. , Providence, 1974,pp. 49-128.

[8] N. Zorboska, Compact composition operatprs on some weighted Hardy spaces, J. Operator Theory **22** (1989), 233-241.

Received: May, 2012