International Mathematical Forum, Vol. 7, 2012, no. 52, 2597 - 2602

On Syndetically Hypercyclic Tuples

Mezban Habibi

Department of Mathematics
Dehdasht Branch, Islamic Azad University, Dehdasht, Iran
P. O. Box 181 40, Lidingo, Stockholm, Sweden
habibi.m@iaudehdasht.ac.ir

Abstract

In this paper we will give some conditions for a tuple of operators or tuple of weighted shifts to be Syndetically Hypercyclic.

Mathematics Subject Classification: 47A16, 47B37

Keywords: Syndetically tuple, Hypercyclic tuple, Hypercyclic vector, Hypercyclicity Criterion, Syndetically Hypercyclic

1 Introduction

Let F be a topological vector space(TVS) and T_1,T_2,...,T_n are continuous mapping on F, and $T = (T_1, T_2, ..., T_n)$ be a tuple of operators T_1,T_2,...,T_n.The tuple T is weakly mixing, if and only if, For any pair of non-empty open subsets U,V in X, and for any syndetic sequences $\{m_{k,1}\}, \{m_{k,2}\}, ..., \{m_{k,n}\}$ with $Sup_k(n_{k+1,j} - n_{k,j}) < \infty$ for $j = 1, 2, ..., n$, then there exist $m_{k,1}, m_{k,2}, ..., m_{k,n}$ such that $T_1^{m_{k,1}} T_2^{m_{k,2}} ... T_n^{m_{k,n}} U \cap V \neq \emptyset$, also, if and only if, it suffices in previous condition, to consider only those sequences $m_{k,1}, m_{k,2}, ..., m_{k,n}$ for which there is some $m_1 \geq 1$, $m_2 \geq 1$,...,$m_n \geq 1$ with $m_{k,1} \in \{m_j, 2m_j\}$ for all k and all j. Reader can see $[1 - -10]$ for some information.

2 Main Results

Let $\mathcal{X}$ be a metrizable and complete topological vector space($F-$space) and $\mathcal{T} = (T_1, T_2, ..., T_n)$ is an n-tuple of operators, then we will let

$$\mathcal{F} = \{{T_1}^{k_1} {T_2}^{k_2} ... {T_n}^{k_n} : k_i \geq 0\}$$

be the semigroup generated by $\mathcal{T}$. For $x \in \mathcal{X}$ we take

$$Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\} = \{{T_1}^{k_1} {T_2}^{k_2} ... {T_n}^{k_n}(x) : k_i \geq 0, i = 1, 2, ..., n\}.$$

The set $Orb(\mathcal{T}, x)$ is called, orbit of vector x under $\mathcal{T}$ and Tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ is called hypercyclic pair if the set $Orb(\mathcal{T}, x)$ is dense in $\mathcal{X}$, that is

$$\overline{Orb(\mathcal{T}, x)} = \overline{\{{T_1}^{k_1}{T_2}^{k_2}...{T_n}^{k_n}(x) : k_i \geq 0, i = 1, 2, ..., n\}} = \mathcal{X}.$$

The tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ is called topologically mixing if for any given open subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{X}$, there exist positive numbers K_1, K_2,...,$_{K_n}$ such that

$${T_1}^{k_{1,i}}{T_2}^{k_{2,i}}...{T_{n,i}}^{k_{n,i}}(\mathcal{U}) \cap \mathcal{V} \neq \phi \quad , \quad \forall k_{j,i} \geq K_i \quad , \quad \forall j = 1, 2, ..., n$$

A sequence of operators $\{T_n\}_{n\geq 0}$ is said to be a hypercyclic sequence on X if there exists some $x \in X$ such that its orbit is dense in X, that is

$$\overline{Orb(\{T_n\}_{n\geq 0}, x)} = \overline{Orb(\{x, T_1x, T_2x, ...\}} = X$$

In this case the vector x is called hypercyclic vector for the sequence $\{T_n\}_{n\geq 0}$. Note that, if $\{T_n\}_{n\geq 0}$ is a hypercyclic sequence of operators on X, then X is necessarily separable. Also note that, the sequence $\{T^n\}$ is a hypercyclic sequence on X, if and only if, the operator T is hypercyclic operator on X. T is said to satisfy the Hypercyclicity Criterion if it satisfies the hypothesis of below theorem.

Theorem 2.1 (The Hypercyclicity Criterion) *Let X be a separable Banach space and $T = (T_1, T_2, ..., T_n)$ is an n-tuple of continuous linear mappings on X. If there exist two dense subsets Y and Z in X, and strictly increasing sequences $\{m_{j,1}\}_{j=1}^{\infty}$, $\{m_{j,2}\}_{j=1}^{\infty}$,...,$\{m_{j,n}\}_{j=1}^{\infty}$ such that:*
1. $T_1^{m_{j,1}}T_2^{m_{j,2}}...T_n^{m_{j,n}} \to 0$ on Y as $j \to \infty$,
2. There exist functions $\{S_j : Z \to X\}$ such that for every $z \in Z$, $S_jz \to 0$, and $T_1^{m_{j,1}}T_2^{m_{j,2}}...T_n^{m_{j,n}}S_jz \to z$,
then T is a hypercyclic n-tuple.

A strictly increasing sequence of positive integers $\{n_k\}_k$ is said to be syndetic sequence, if $Sup_k(n_{k+1} - n_k) < \infty$.
A tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ on a space $\mathcal{X}$ is called syndetically hypercyclic if for any syndetic sequences of positive integers

$$\{m_{k,1}\}_k, \{m_{k,2}\}_k, ..., \{m_{k,n}\}_k$$

the sequence

$$\{T_1^{m_{k,1}}T_2^{m_{k,2}}...T_n^{m_{k,n}}\}_k$$

is hypercyclic, in other hand, there is $x \in X$ such that $\{T^{n_k}x : k \geq 0\}$ is dens in X, that is,

$$\overline{\{T_1^{m_{k,1}}T_2^{m_{k,2}}...T_n^{m_{k,n}}(x)\}} = X.$$

Given continuous linear operators T_1,T_2,...,T_n that is $T_1, T_2, ..., T_n \in L(X)$, defined on a separable F-space X, also suppose that $T = (T_1, T_2, ..., T_n)$ be a tuple of operators T_1,T_2,...,T_n, Then T satisfies the Hypercyclicity Criterion if and only if for any strictly increasing sequences of positive integers $\{m_{k,1}\}_k, \{n_{k,2}\}_k, ..., \{m_{k,n}\}_k$ such that

$$Sup_k(m_{k+1,j} - m_{k,j}) < \infty$$

for all j, then the sequence $\{T_1^{m_{k,1}}T_2^{m_{k,2}}...T_n^{m_{k,n}}\}_k$ is hypercyclic. Also, for each hypercyclic vector $x \in X$ of T, there exists two strictly increasing sequence $\{m_k\}_k$,$\{n_k\}_k$ such that

$$Sup_k(m_{k+1,j} - n_{k,j}) < \infty$$

for all j, and $\{T_1^{m_{k,1}}T_2^{m_{k,2}}...T_n^{m_{k,n}}\}_k$ is somewhere dense, but not dense in X, That is, the tuple $T = (T_1, T_2, ..., T_n)$ and the sequence $\{T_1^{m_{k,1}}T_2^{m_{k,2}}...T_n^{m_{k,n}}\}_k$ do not share the same hypercyclic vectors. Let F be a Frechet space and $T_1, T_2, ..., T_n$ are bounded linear operators on F, and $T = (T_1, T_2, ..., T_n)$ be a tuple of operators $T_1, T_2, ..., T_n$. The space F is called topologically mixing if for any given open sets U and V, there exist positive numbers $M_1, M_{2,...,M_n}$ such that

$$T_1^{m_{i,1}}T_2^{m_{i,2}}...T_n^{m_{i,n}}(U) \cap V \neq \phi \quad , \quad \forall m_{i,j} \geq M_j \quad , \quad i = 1, 2, ..., n$$

Notice that, If the tuple T satisfies the hypercyclic criterion for syndetic sequences, then T is topologically mixing tuple on space F. Let V be a topological vector space(TVS) and $T_1, T_2, ..., T_n$ are bounded linear operators on V, and $T = (T_1, T_2, ..., T_n)$ be a tuple of operators $T_1, T_2, ..., T_n$. The tuple T is called weakly mixing if

$$T \times T \times ... \times T : X \times X \times ... \times X \to X \times X \times ... \times X$$

is topologically transitive.

Theorem 2.2 *Let X be a topological vector space(TVS) and $T_1, T_2, ..., T_n$ are continuous mapping on X, and $T = (T_1, T_2, ..., T_n)$ be a tuple of operators $T_1, T_2, ..., T_n$. Then the following are equivalent:*
(i). T is weakly mixing.
(ii). For any pair of non-empty open subsets U,V in X, and for any syndetic sequences $\{m_{k,1}\}, \{m_{k,2}\}, ..., \{m_{k,n}\}$, there exist $m'_{k,1}, m'_{k,2}, ..., m'_{k,n}$ such that

$$T_1^{m'_{k,1}}T_2^{m'_{k,2}}...T_n^{m'_{k,n}}(U) \cap (V) \neq \emptyset$$

(iii). It suffices in (ii) to consider only those sequences $\{m_{k,1}\},\{m_{k,2}\},...,\{m_{k,n}\}$ for which there is some $m_1 \geq 1$, $m_2 \geq 1$,...,$m_n \geq 1$ with

$$m_{k,j} \in \{m_j, 2m_j\}$$

for all k and for all j.

proof $(i) \rightarrow (ii)$. Given $\{m_{k,1}\},\{m_{k,2}\},...,\{m_{k,n}\}$ and U,V satisfying the hypothesis of condition (ii), take

$$m_j = Sup_k\{m_{k+1,j} - m_{k,j}\}$$

for all j and the $n-$product map

$$\overbrace{T \times T ... \times T}^{n-tims} : \overbrace{X \times X ... \times X}^{n-tims} \rightarrow \overbrace{X \times X ... \times X}^{n-tims}$$

is transitive, Then there is $m_{k',1}, m_{k',2},...,m_{k',n}$ in N such that

$$(T_1^{m_{k',1}} T_2^{m_{k',2}} ... T_n^{m_{k',n}}(U)) \bigcap ((T_1^{m_{k'',1}})^{-1} (T_2^{m_{k'',2}})^{-1} ... (T_n^{m_{k'',n}})^{-1}(V)) \neq \emptyset$$

$$\forall m_{k'',1} = 1,2,...,m, \forall m_{k'',2} = 1,2,...,m,..., \forall m_{k'',n} = 1,2,...,m$$

so

$$(T_1^{m_{k',1}+m_{k'',1}} T_2^{m_{k',2}+m_{k'',2}} ... T_n^{m_{k',n}+m_{k'',n}}(U)) \bigcap (V)) \neq \emptyset$$

$$\forall m_{k'',1} = 1,2,...,m, \forall m_{k'',2} = 1,2,...,m,..., \forall m_{k'',n} = 1,2,...,m$$

By the assumption on $\{m_{k,1}\},\{m_{k,2}\},...,\{m_{k,n}\}$,for all j, we have

$$\{m_{k,j} : k \in N\} \cap \{n+1, n+2, ..., n+m_j\} \neq \emptyset$$

If for all j we select $m'_{k,j} \in \{m_{k,j} : k \in N\} \cap \{n+1, n+2, ..., n+m_j\}$ then we have $T_1^{m'_{k,1}} T_2^{m'_{k,2}} ... T_n^{m'_{k,n}}(U) \cap (V) \neq \emptyset$, by this the proof of $(i) \rightarrow (ii)$ is completed.

The case $(ii) \rightarrow (iii)$ is trivial.

Case $(iii) \rightarrow (i)$. Suppose that U, V_1, V_2 are non-empty open subsets of X, then there are $\{m_{k,1}\},\{m_{k,2}\},...,\{m_{k,n}\}$ in N such that

$$T_1^{m_{k,1}} T_2^{m_{k,2}} ... T_n^{m_{k,n}} U \cap V_1 \neq \emptyset$$

$$T_1^{m_{k,1}} T_2^{m_{k,2}} ... T_n^{m_{k,n}} U \cap V_2 \neq \emptyset.$$

This will imply that T is weakly mixing. Since (iii) is satisfied, then we can take $\{m_{k,1}\},\{m_{k,2}\},...,\{m_{k,n}\}$ in N such that

$$T_1^{m_{k,1}} T_2^{m_{k,2}} ... T_n^{m_{k,n}} V_1 \cap V_2 \neq \emptyset$$

By continuity, we can find $\widetilde{V_1} \subset V_1$ open and non-empty such that

$$T_1^{m_{k,1}} T_2^{m_{k,2}} ... T_n^{m_{k,n}} \widetilde{V_1} \subset V_2.$$

Also there exist some $m_{k',1}, m_{k',2}, ..., m_{k',n}$ in N such that

$$T_1^{m_{k',1}+\eta_1} T_2^{m_{k',2}+\eta_2} ... T_n^{m_{k',n}+\eta_n} U \subset \widetilde{V_1}$$

for $\eta_j = 0, m_j$ we take $m_{k,j} = m_{k',j} + \eta_j$, for all j, indeed we find strictly increasing sequences of positive integers $m_{k,1}, m_{k,2}, ..., m_{k,n}$ such that

$$m_{k,j} \in \{m_j, 2m_j\}$$

for all j,and

$$T_1^{m_{k,1}} T_2^{m_{k,2}} ... T_n^{m_{k,n}} U \cap \widetilde{V_1} = \emptyset, \forall k \in N$$

Now we have

$$T_1^{m_{k',1}+\eta_1} T_2^{m_{k',2}+\eta_2} ... T_n^{m_{k',n}+\eta_n} U \bigcap \widetilde{V_1} \neq \emptyset$$

So the set

$$\emptyset \neq T_1^{m_{k,1}} T_2^{m_{k,2}} ... T_n^{m_{k,n}} (T_1^{m'_{k,1}} T_2^{m'_{k,2}} ... T_n^{m'_{k,n}} U \cap \widetilde{V_1})$$

is a subset of

$$(T_1^{m_{k',1}+\eta_1} T_2^{m_{k',2}+\eta_2} ... T_n^{m_{k',n}+\eta_n} U) \cap (T_1^{m_{k,1}} T_2^{m_{k,2}} ... T_n^{m_{k,n}} \widetilde{V_1})$$

then we have

$$T_1^{m_{k,1}} T_2^{m_{k,2}} ... T_n^{m_{k,n}} (U) \bigcap (V_1) \neq \phi$$

and similarly

$$T_1^{m_{k,1}} T_2^{m_{k,2}} ... T_n^{m_{k,n}} (U) \bigcap (V_2) \neq \phi$$

now, this is the end of proof.

References

[1] J. Bes and A. Peris, Hereditarily hypercylic operators, *Jour. Func. Anal.* , **1** (167) (1999), 94-112.

[2] M. Habibi, n-Tuples and chaoticity, *Int. Journal of Math. Analysis*, **6** (14) (2012), 651-657 .

[3] M. Habibi, ∞-Tuples of Bounded Linear Operators on Banach Space, *Int. Math. Forum*, **7** (18) (2012), 861-866.

[4] M. Habibi and F. Safari, n-Tuples and Epsilon Hypercyclicity, *Far East Jour. of Math. Sci.* , **47** (2) (2010), 219-223.

[5] M. Habibi and B. Yousefi, Conditions for a tuple of operators to be topologically mixing, *Int. Jour. of App. Math.* , **23**(6) (2010), 973-976.

[6] A. Peris and L. Saldivia, Syndentically hypercyclic operators, *Integral Equations Operator Theory* , **51**, No. 2 (2005) 275-281.

[7] H. N. Salas, Hypercyclic weighted shifts, *Trans. Amer. Math. Soc.* , 347 (1995), 993-1004.

[8] B. Yousefi and M. Habibi, Syndetically Hypercyclic Pairs, *Int. Math. Forum* , **5** (66) (2010), 3267 - 3272.

[9] B. Yousefi and M. Habibi, Hereditarily Hypercyclic Pairs, *Int. Jour. of App. Math.* , **24**(2) (2011)), 245-249.

[10] B. Yousefi and M. Habibi, Hypercyclicity Criterion for a Pair of Weighted Composition Operators, *Int. Jour. of App. Math.* , **24** (2) (2011), 215-219.

Received: May, 2012