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Generalized Weak Structures

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Abstract

In this work we define generalized weak structures which naturally generalize minimal structures, generalized topologies and weak structures recently proposed in the literature. Moreover, we study some properties of the interior, the closure and other concepts in this new context.

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Introduction

The study of more general structures than that of topological space has taken several directions over the past fifteen years. In 1996, Maki ([3]) studied *minimal structures*, or shortly *m-structures*, on a set X, i.e., collections for subsets of X containing the empty set and X, with no other restriction. Since 1997, Császár has studied topological notions in collections which are closed under unions ([1]). They constitute the well-known *generalized topologies*. As a natural generalization of the above-mentioned structures, in 2011 this author ([2]) also introduces the *weak structures*, which are collections of subsets of X containing the empty set. In addition, he defines interior and closure within this new context and shows important properties of these operations.

In this paper we define *generalized weak structures* as an extension of Császár's *weak structures*. For them, we introduce interior, closure and other related notions. We certainly show that many properties of these "familiar" notions remain valid under our more general assumptions.

1 The $\mathfrak{g}-$interior and $\mathfrak{g}-$closure

In this section we generalize all the results presented in [2] and particularly well-known results in generalized topological spaces ([1]) and minimal structures ([3] and [4]).

Definition 1 *A generalized weak structure (GWS) on the nonempty set X, is a nonempty class $\mathfrak{g}$ of subsets of X.*

If $\mathfrak{g}$ is a generalized weak structure on X, then each element of $\mathfrak{g}$ is said to be $\mathfrak{g}-$open and the complement (in X) of a $\mathfrak{g}-$open set is a $\mathfrak{g}-$closed set.

It is clear that each generalized topology, each minimal structure and each weak structure is a GWS.

Definition 2 *Let $\mathfrak{g}$ be a GWS on X and $A \subseteq X$. The $\mathfrak{g}$ − interior and the $\mathfrak{g}$ − closure of A are defined by $i_{\mathfrak{g}}(A) = \cup\{U : U \subseteq A, U \in \mathfrak{g}\}$ and $c_{\mathfrak{g}}(A) = \cap\{F : A \subseteq F, F^c \in \mathfrak{g}\}$ respectively.*

We note that in certain cases of GWS's, it could happen that $c_{\mathfrak{g}}(\emptyset) \neq \emptyset$ and $i_{\mathfrak{g}}(X) \neq X$.

In the following result we show the main properties of the $\mathfrak{g}-$interior and the $\mathfrak{g}-$closure. In particular, we prove that the operators $i_{\mathfrak{g}}$ and $c_{\mathfrak{g}}$ are idempotent and monotonic.

Proposition 3 *Let $\mathfrak{g}$ be a GWS on X and $A, B \subseteq X$. The following properties hold:*

1. $i_{\mathfrak{g}}(A) \subseteq A$.
2. *If* $A \in \mathfrak{g}$, *then* $i_{\mathfrak{g}}(A) = A$.
3. *If* $A \subseteq B$, *then* $i_{\mathfrak{g}}(A) \subseteq i_{\mathfrak{g}}(B)$.
4. $i_{\mathfrak{g}}(i_{\mathfrak{g}}(A)) = i_{\mathfrak{g}}(A)$.
5. $A \subseteq c_{\mathfrak{g}}(A)$.
6. *If* $A^c \in \mathfrak{g}$, *then* $c_{\mathfrak{g}}(A) = A$.
7. *If* $A \subseteq B$, *then* $c_{\mathfrak{g}}(A) \subseteq c_{\mathfrak{g}}(B)$.
8. $c_{\mathfrak{g}}(c_{\mathfrak{g}}(A)) = c_{\mathfrak{g}}(A)$.

Proof. Items 1., 3., 5. and 7. are obvious.

2. If $A \in \mathfrak{g}$, then $A \subseteq \cup\{U : U \subseteq A, U \in \mathfrak{g}\} = i_{\mathfrak{g}}(A)$. Thus by 1. the result follows.

4. By 1. $i_{\mathfrak{g}}(i_{\mathfrak{g}}(A)) \subseteq i_{\mathfrak{g}}(A)$. On the other hand, if $x \in i_{\mathfrak{g}}(A)$ then there exists $U \in \mathfrak{g}$ such that $U \subseteq A$. So, $x \in U \subseteq i_{\mathfrak{g}}(A)$ which implies that $x \in i_{\mathfrak{g}}(i_{\mathfrak{g}}(A)$.

6. It is enough to observe that if $A^c \in \mathfrak{g}$ then A is an element of the collection $\{F : A \subseteq F, F^c \in \mathfrak{g}\}$.

8. By 5. $c_{\mathfrak{g}}(A) \subseteq c_{\mathfrak{g}}(c_{\mathfrak{g}}(A))$. For the other inclusion is enough to note that $\{F : A \subseteq F, F^c \in \mathfrak{g}\} \subseteq \{B : c_{\mathfrak{g}}(A) \subseteq B, B^c \in \mathfrak{g}\}$. ■

The converse of Proposition 3 (2, 6) is not true in general. However, it is evident that, if $\mathfrak{g}$ is closed under unions, then the result is guaranteed.

Example 4 Consider $X = \{a, b, c, d\}$ with $\mathfrak{g} = \{\{a\}, \{b\}, \{c\}, \{d\}\}$. For $A = \{a, c\}$ we have $i_{\mathfrak{g}}(A) = A = c_{\mathfrak{g}}(A)$, but the set A is neither $\mathfrak{g}-$open nor $\mathfrak{g}-$closed.

The well-known relationship between the interior and closure remains valid in our case. Also, the classical characterization of these concepts in terms of its elements can be extended to GWS's.

Proposition 5 *Let $\mathfrak{g}$ be a GWS on X, $A \subseteq X$ and $x \in X$. The following statements hold:*

1. $i_{\mathfrak{g}}(A)^c = c_{\mathfrak{g}}(A^c)$.
2. $i_{\mathfrak{g}}(A^c) = c_{\mathfrak{g}}(A)^c$.
3. $x \in i_{\mathfrak{g}}(A)$, *iff, there exists* $U \in \mathfrak{g}$ *such that* $x \in U \subseteq A$.
4. $x \in c_{\mathfrak{g}}(A)$, *iff,* $U \cap A \neq \emptyset$ *for all* $U \in \mathfrak{g}$ *with* $x \in U$.

Proof. 1. $i_{\mathfrak{g}}(A)^c = \cap\{U^c : A^c \subseteq U^c, (U^c)^c \in \mathfrak{g}\} = c_{\mathfrak{g}}(A^c)$.

2. It is enough to replace A by A^c in item 1.

3. It follows easily from Definition 2.

4. Suppose that there exists $U \in \mathfrak{g}$ which contains x and $U \cap A = \emptyset$. Then $A \subseteq U^c$ and thus $x \notin c_{\mathfrak{g}}(A)$. Conversely, if $x \notin c_{\mathfrak{g}}(A)$ then there exists a $\mathfrak{g}-$closed set F with $A \subseteq F$ and $x \notin F$. So, $x \in F^c \in \mathfrak{g}$ and $F^c \cap A = \emptyset$. ■

Given a GWS $\mathfrak{g}$ on X it is possible to consider functions from $P(X)$ to $P(X)$ with the form $\phi_1 \circ \ldots \circ \phi_k$, where $\phi_k = i_{\mathfrak{g}}$ or $\phi_k = c_{\mathfrak{g}}$ for all k. For example, $i_{\mathfrak{g}}i_{\mathfrak{g}} = i_{\mathfrak{g}}$ and $c_{\mathfrak{g}}c_{\mathfrak{g}} = c_{\mathfrak{g}}$ as we have shown.

Proposition 6 *Let $\mathfrak{g}$ be a GWS on X and $A \subseteq X$. Then* $i_{\mathfrak{g}}c_{\mathfrak{g}}i_{\mathfrak{g}}c_{\mathfrak{g}} = i_{\mathfrak{g}}c_{\mathfrak{g}}$ *and* $c_{\mathfrak{g}}i_{\mathfrak{g}}c_{\mathfrak{g}}i_{\mathfrak{g}} = c_{\mathfrak{g}}i_{\mathfrak{g}}$.

Proof. Let $A \in P(X)$. It is clear that $i_{\mathfrak{g}}c_{\mathfrak{g}}(A) \subseteq c_{\mathfrak{g}}(A)$ and since $i_{\mathfrak{g}}$ and $c_{\mathfrak{g}}$ are idempotent and monotonic we have $c_{\mathfrak{g}}i_{\mathfrak{g}}c_{\mathfrak{g}}(A) \subseteq c_{\mathfrak{g}}(A)$ and $i_{\mathfrak{g}}c_{\mathfrak{g}}i_{\mathfrak{g}}c_{\mathfrak{g}}(A) \subseteq i_{\mathfrak{g}}c_{\mathfrak{g}}(A)$. On the other hand, $i_{\mathfrak{g}}c_{\mathfrak{g}}(A) \subseteq c_{\mathfrak{g}}i_{\mathfrak{g}}c_{\mathfrak{g}}(A)$ which implies that $i_{\mathfrak{g}}c_{\mathfrak{g}}(A) \subseteq i_{\mathfrak{g}}c_{\mathfrak{g}}i_{\mathfrak{g}}c_{\mathfrak{g}}(A)$. The remaining part is analogous. ■

In a similar way to [2] we can obtain several structures. If $\mathfrak{g}$ is a GWS on X and $A \subseteq X$ we can define: $A \in \alpha(\mathfrak{g})$ iff $A \subseteq i_{\mathfrak{g}}c_{\mathfrak{g}}i_{\mathfrak{g}}(A)$, $A \in \sigma(\mathfrak{g})$ iff $A \subseteq c_{\mathfrak{g}}i_{\mathfrak{g}}(A)$, $A \in \pi(\mathfrak{g})$ iff $A \subseteq i_{\mathfrak{g}}c_{\mathfrak{g}}(A)$, $A \in \rho(\mathfrak{g})$ iff $A \subseteq c_{\mathfrak{g}}i_{\mathfrak{g}}(A) \cup i_{\mathfrak{g}}c_{\mathfrak{g}}(A)$ and $A \in \beta(\mathfrak{g})$ iff $A \subseteq c_{\mathfrak{g}}i_{\mathfrak{g}}c_{\mathfrak{g}}(A)$.

In the following, we shall write i for $i_{\mathfrak{g}}$ and c for $c_{\mathfrak{g}}$ for a given GWS $\mathfrak{g}$.

Proposition 7 *Let $\mathfrak{g}$ be a GWS on X and $A \subseteq X$. Then, $\mathfrak{g} \subseteq \alpha(\mathfrak{g}) \subseteq \sigma(\mathfrak{g}) \subseteq \rho(\mathfrak{g}) \subseteq \beta(\mathfrak{g})$ y $\alpha(\mathfrak{g}) \subseteq \pi(\mathfrak{g}) \subseteq \rho(\mathfrak{g})$.*

Proof. If $A \in \mathfrak{g}$, then $A = i(A) \subseteq ci(A)$. Since i is monotonic and idempotent we have that $A = i(A) \subseteq ici(A)$. Thus $A \in \alpha(\mathfrak{g})$.

If $A \in \alpha(\mathfrak{g})$, then $A \subseteq ici(A) \subseteq ci(A)$. Thus $A \in \sigma(\mathfrak{g})$.

If $A \in \sigma(\mathfrak{g})$, then $A \subseteq ci(A) \subseteq ci(A) \cup ic(A)$. Thus $A \in \rho(\mathfrak{g})$.

If $A \in \rho(\mathfrak{g})$, then either $A \in \sigma(\mathfrak{g}) \subseteq \beta(\mathfrak{g})$ or $A \in ic(A)$. So, in any case $c(A) \subseteq cic(A)$ implying $A \subseteq cic(A)$. Thus $A \in \beta(A)$.

We know that $i(A) \subseteq A$ and since i and c are monotonic we obtain that $ici(A) \subseteq ic(A)$. So, if $A \in \alpha(\mathfrak{g})$ then $A \subseteq ici(A) \subseteq ic(A)$. Thus $A \in \pi(\mathfrak{g})$.

It is evident that $A \in \pi(\mathfrak{g})$ implies $A \subseteq ic(A) \subseteq ci(A) \cup ic(A)$. Thus $A \in \rho(\mathfrak{g})$. ■

2 Other elementary concepts

Definition 8 *Let $\mathfrak{g}$ be a GWS on X, $A \subseteq X$. The $\mathfrak{g}-$derived set of A, $d_{\mathfrak{g}}(A)$, is the set of all points $x \in X$ such that every $U \in \mathfrak{g}$ containing x satisfies $(U \setminus \{x\}) \cap A \neq \emptyset$.*

In the following proposition we prove that each $\mathfrak{g}-$closed set contains its $\mathfrak{g}-$derived set. Moreover, we use this last concept to characterize the $\mathfrak{g}-$closure.

Proposition 9 *Let $\mathfrak{g}$ be a GWS on X and $A, B \subseteq X$. The following statements hold:*

1. *If $A \subseteq B$, then $d_{\mathfrak{g}}(A) \subseteq d_{\mathfrak{g}}(B)$.*

2. *$d_{\mathfrak{g}}(A) \subseteq c_{\mathfrak{g}}(A)$.*

3. *If $A^c \in \mathfrak{g}$, then $d_{\mathfrak{g}}(A) \subseteq A$.*

4. *$c_{\mathfrak{g}}(A) = A \cup d_{\mathfrak{g}}(A)$.*

Proof. 1. If $x \in d_{\mathfrak{g}}(A)$ and $x \in U \in \mathfrak{g}$ then $\emptyset \neq (U \setminus \{x\}) \cap A \subseteq (U \setminus \{x\}) \cap B$. Thus $x \in c_{\mathfrak{g}}(B)$.

2. It is analogous to item 1.

3. Suppose that $x \in d_{\mathfrak{g}}(A)$ with $A^c \in \mathfrak{g}$. If $x \notin A$, then $x \in A^c$ and $A \cap A^c = \emptyset$ which implies that $x \notin d_{\mathfrak{g}}(A)$. Thus $x \in A$ and the result follows.

4. The inclusion $A \cup d_{\mathfrak{g}}(A) \subseteq c_{\mathfrak{g}}(A)$ is clear. If $x \in c_{\mathfrak{g}}(A)$ and $x \notin A$, then each $\mathfrak{g}-$open set U with $x \in U$ satisfies $U \cap A \neq \emptyset$. Then $(U \setminus \{x\}) \cap A \neq \emptyset$ because $x \notin A$. Hence, $x \in d_{\mathfrak{g}}(A)$ and the other inclusion holds. ■

The converse of Proposition 9(3) is not true in general.

Example 10 Consider $\mathbb{R}$ with $\mathfrak{g} = \{[a,b] : a, b \in \mathbb{R}\}$. If $A = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$, then $d_{\mathfrak{g}}(A) = \{0\} \subseteq A$. However, A is not a $\mathfrak{g}-$closed set because $A^c \notin \mathfrak{g}$.

Similar to topological spaces we can define other concepts such as the boundary and exterior of a set.

Definition 11 *Let $\mathfrak{g}$ be a GWS on X and $A \subseteq X$. The $\mathfrak{g}-$exterior and $\mathfrak{g}-$boundary of A are defined as $e_{\mathfrak{g}}(A) = i_{\mathfrak{g}}(A^c)$ and $b(A) = c_{\mathfrak{g}}(A) \cap c_{\mathfrak{g}}(A^c)$ respectively.*

Proposition 12 *Let $\mathfrak{g}$ be a GWS on X and $A, B \subseteq X$. The following affirmations hold:*

1. *$e_{\mathfrak{g}}(A) \subseteq A^c$.*
2. *If $A^c \in \mathfrak{g}$, then $e_{\mathfrak{g}}(A) = A^c$.*
3. *If $A \subseteq B$, then $e_{\mathfrak{g}}(B) \subseteq e_{\mathfrak{g}}(A)$.*
4. *$b_{\mathfrak{g}}(A^c) = b_{\mathfrak{g}}(A)$.*
5. *If $A \in \mathfrak{g}$, then $A \cap b_{\mathfrak{g}}(A) = \emptyset$.*
6. *If $A^c \in \mathfrak{g}$, then $b_{\mathfrak{g}}(A) \subseteq A$.*

We finish this section by presenting other properties that can be easily extended to generalized weak structures.

Proposition 13 *Let $\mathfrak{g}$ be a GWS on X and $A \subseteq X$. The following statements hold:*

1. *$c_{\mathfrak{g}}(A) \setminus i_{\mathfrak{g}}(A) = b_{\mathfrak{g}}(A)$.*
2. *$X \setminus b_{\mathfrak{g}}(A) = i_{\mathfrak{g}}(A) \cup e_{\mathfrak{g}}(A)$.*
3. *$A \setminus b_{\mathfrak{g}}(A) = i_{\mathfrak{g}}(A)$.*
4. *$b_{\mathfrak{g}}(A) \cup A = c_{\mathfrak{g}}(A)$.*
5. *$b_{\mathfrak{g}}(A) \cup i_{\mathfrak{g}}(A) = c_{\mathfrak{g}}(A)$.*
6. *$X = i_{\mathfrak{g}}(A) \sqcup b_{\mathfrak{g}}(A) \sqcup e_{\mathfrak{g}}(A)$, where "$\sqcup$" denotes disjoint union.*

3 $\mathfrak{g}-$subspaces

Definition 14 *Let $\mathfrak{g}$ be a GWS on X and $A \subseteq X$. The GWS on A associated to $\mathfrak{g}$ is defined by the collection $\mathfrak{g}_A = \{G \cap A : G \in \mathfrak{g}\}$. In this case A is called a $\mathfrak{g}-$subspace of X.*

Note that if A is a $\mathfrak{g}-$subspace of X then $K \subseteq A$ is $\mathfrak{g}_A-$closed, if and only if, $K = F \cap A$ for some $\mathfrak{g}-$closed set F.

Proposition 15 *Let A be a $\mathfrak{g}-$subspace of X and $B \subseteq A$. The following statements hold:*

1. $d_{\mathfrak{g}_A}(B) = d_{\mathfrak{g}}(B) \cap A$.
2. $c_{\mathfrak{g}_A}(B) = c_{\mathfrak{g}}(B) \cap A$.
3. $i_{\mathfrak{g}_A}(B) \supseteq i_{\mathfrak{g}}(B) \cap A$.
4. $b_{\mathfrak{g}_A}(B) \subseteq b_{\mathfrak{g}}(B) \cap A$.

Proof. 1. Let $x \in d_{\mathfrak{g}_A}(B)$ and $H \in \mathfrak{g}$ such that $x \in H$. Then $H \cap A \in \mathfrak{g}_A$ and thus $((H \cap A) \setminus \{x\}) \cap B \neq \emptyset$ which implies that $(H \setminus \{x\}) \cap B \neq \emptyset$. Therefore, $x \in d_{\mathfrak{g}}(B)$ and we have the inclusion $d_{\mathfrak{g}_A}(B) \subseteq d_{\mathfrak{g}}(B) \cap A$. Let $x \in d_{\mathfrak{g}}(B) \cap A$ and $G \in \mathfrak{g}_A$ such that $x \in G$. Since $G = H \cap A$ with $H \in \mathfrak{g}$ we have that $(G \setminus \{x\}) \cap B = (H \setminus \{x\}) \cap B \neq \emptyset$. Thus $x \in d_{\mathfrak{g}_A}(B)$ and the other inclusion holds.

2. $c_{\mathfrak{g}_A}(B) = B \cup d_{\mathfrak{g}_A}(B) = B \cup (d_{\mathfrak{g}}(B) \cap A) = (B \cup d_{\mathfrak{g}}(B)) \cap (B \cup A) = c_{\mathfrak{g}}(B) \cap A$.

3. If $x \in i_{\mathfrak{g}}(B) \cap A$, then $x \in A$ and there exists $G \in \mathfrak{g}$ such that $x \in G \subseteq B$. So, $x \in A \cap G \subseteq B$ and consequently $x \in i_{\mathfrak{g}_A}(B)$.

4. It is consequence of item 2. ■

If $\mathfrak{g}$ is a GWS on X and $B \subseteq A \subseteq X$, the set B can be regarded as a $\mathfrak{g}-$subspace of X with the GWS $\mathfrak{g}_B$ on B, or as a $\mathfrak{g}_A-$subspace of A with the GWS $(\mathfrak{g}_A)_B$ on B.

Proposition 16 *Let $\mathfrak{g}$ be a GWS on X and $B \subseteq A \subseteq X$. Then $\mathfrak{g}_B = (\mathfrak{g}_A)_B$.*

Proof. If $G \in \mathfrak{g}_B$, then there exists $H \in \mathfrak{g}$ such that $G = H \cap B$. So, $G = (H \cap A) \cap B \in (m_A)_B$ and hence $\mathfrak{g}_B \subseteq (\mathfrak{g}_A)_B$. The other inclusion is analogous. ■

Given a GWS $\mathfrak{g}$ on X the property $\mathcal{P}$ on X is called hereditary if every $\mathfrak{g}-$subspace of X satisfies $\mathcal{P}$.

Example 17 *Suppose that $\mathfrak{g}$ is a GWS on X and $A \subseteq X$. If $\mathfrak{g}$ is closed under unions, then so is $\mathfrak{g}_A$. In fact, if $\{H_i \cap A\}_{i \in I}$ is a collection of elements of $\mathfrak{g}_A$, then $\cup_{i \in I}(H_i \cap A) = (\cup_{i \in I} H_i) \cap A \in \mathfrak{g}_A$ since $\cup_{i \in I} H_i \in \mathfrak{g}$.*

References

[1] Á. Császár, Generalized open sets, Acta Math. Hungar., 75 (1997), 65–87.

[2] Á. Császár, Weak structures, Acta Math. Hungar., 131 (2011), 193–195.

[3] H. Maki, On generalizing semi-open and preopen sets, Report for Meeting on Topological Spaces and its Applications, Yatsushiro College of Technology, (1996), 13–18.

[4] V. Popa and T. Noiri, On the definitions of some generalized forms of continuity under minimal conditions, Mem. Fac. Sci. Kochi Univ. Ser. Math., 22 (2001), 9–18.

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