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# A Hessian Domain Constructed with a Foliation by 1-Conformally Flat Statistical Manifolds

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**Abstract**

A Hessian domain is a flat statistical manifold, and its level surfaces are 1-conformally flat statistical submanifolds. In this paper we show conditions for that 1-conformally flat statistical leaves of a foliation can be realized as level surfaces of their common Hessian domain conversely.

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**Keywords:** Hessian domain, potential function, level surface, foliation, statistical manifold

## 1 Introduction

Let $\varphi$ be a function on a domain $\Omega$ in a real affine space $\mathbf{A}^{n+1}$. Denoting by $D$ the canonical flat affine connection on $\mathbf{A}^{n+1}$, we set $g = Dd\varphi$ and suppose that $g$ is non-degenerate. Then a Hessian domain $(\Omega, D, g)$ is a flat statistical manifold [8].

Kurose defined $\alpha$-conformal equivalence and $\alpha$-conformal flatness of statistical manifolds [4]. In [9] we proved that $n$-dimensional level surfaces of $\varphi$ are 1-conformally flat statistical submanifolds of $(\Omega, D, g)$. In addition we show properties of foliations on Hessian domains with respect to statistical submanifolds in [10]. Hao and Shima studied level surfaces on Hessian domains deeply in [2] [7]. However they studied foliations and statistical submanifolds for given Hessian domains. We see few results of the realization of statistical manifolds on Hessian domains. In [9] we show that a 1-conformally flat statistical manifold can be locally realized as a submanifold of a flat statistical manifold, constructing a level surface of a Hessian domain. However we proved realization of only "a" 1-conformally flat statistical manifold. In this paper we give conditions for realization of many 1-conformally flat statistical manifolds as level surfaces of their common Hessian domain.

In section 2 we recall properties of Hessian domains, statistical manifolds and affine differential geometry. In section 3 we show necessary conditions for a Hessian domain and prepare to prove the main theorem. In section 4 we prove the main theorem on realization of 1-conformally flat statistical leaves.

## 2 Hessian domains and Statistical manifolds

Let $D$ and $\{x^1, \ldots, x^{n+1}\}$ be the canonical flat affine connection and the canonical affine coordinate system on $\mathbf{A}^{n+1}$, i.e., $Ddx^i = 0$. If the Hessian $Dd\varphi = \sum_{i,j}(\partial^2\varphi/\partial x^i \partial x^j)dx^i dx^j$ is non-degenerate for a function $\varphi$ on a domain $\Omega$ in $\mathbf{A}^{n+1}$, we call $(\Omega, D, g = Dd\varphi)$ a Hessian domain. For a torsion-free affine connection $\nabla$ and a pseudo-Riemannian metric $h$ on a manifold $N$, the triple $(N, \nabla, h)$ is called a statistical manifold if $\nabla h$ is symmetric. If the curvature tensor $R$ of $\nabla$ vanishes, $(N, \nabla, h)$ is said to be flat. A Hessian domain $(\Omega, D, g = Dd\varphi)$ is a flat statistical manifold. Conversely, a flat statistical manifold is locally a Hessian domain [1][8].

For a statistical manifold $(N, \nabla, h)$, let $\nabla'$ be an affine connection on $N$ such that

$$Xh(Y, Z) = h(\nabla_X Y, Z) + h(Y, \nabla'_X Z), \quad \text{for } X, Y \text{ and } Z \in TN,$$

where $TN$ is the set of all tangent vector fields on $N$. The affine connection $\nabla'$ is torsion free, and $\nabla' h$ symmetric. Then $\nabla'$ is called the dual connection of $\nabla$, the triple $(N, \nabla', h)$ the dual statistical manifold of $(N, \nabla, h)$, respectively.

Let $\mathbf{A}^*_{n+1}$ and $\{x^*_1, \ldots, x^*_{n+1}\}$ be the dual affine space of $\mathbf{A}^{n+1}$ and the dual affine coordinate system of $\{x^1, \ldots, x^{n+1}\}$, respectively. We define the gradient mapping $\iota$ from $\Omega$ to $\mathbf{A}^*_{n+1}$ by

$$x^*_i \circ \iota = -\frac{\partial \varphi}{\partial x^i},$$

and a flat affine connection $D'$ on $\Omega$ by

$$\iota_*(D'_X Y) = D^*_X \iota_*(Y) \quad \text{for} \quad X, Y \in T\Omega,$$

where $D^*_X \iota_*(Y)$ is covariant derivative along $\iota$ induced by the canonical flat affine connection $D^*$ on $\mathbf{A}^*_{n+1}$. Then $(\Omega, D', g)$ is the dual statistical manifold of $(\Omega, D, g)$.

For $\alpha \in \mathbf{R}$, statistical manifolds $(N, \nabla, h)$ and $(N, \bar{\nabla}, \bar{h})$ are said to be $\alpha$-conformally equivalent if there exists a function $\phi$ on $N$ such that

$$\bar{h}(X, Y) = e^{\phi} h(X, Y),$$

$$h(\bar{\nabla}_X Y, Z) = h(\nabla_X Y, Z) - \frac{1+\alpha}{2} d\phi(Z) h(X,Y) + \frac{1-\alpha}{2}\{d\phi(X)h(Y,Z) + d\phi(Y)h(X,Z)\}$$

for $X, Y$ and $Z \in TN$. A statistical manifold $(N, \nabla, h)$ is called $\alpha$-conformally flat if $(N, \nabla, h)$ is locally $\alpha$-conformally equivalent to a flat statistical manifold. Statistical manifolds $(N, \nabla, h)$ and $(N, \bar{\nabla}, \bar{h})$ are $\alpha$-conformally equivalent if and only if the dual statistical manifolds $(N, \nabla', h)$ and $(N, \bar{\nabla}', \bar{h})$ are $(-\alpha)$-conformally equivalent. Especially, a statistical manifold $(N, \nabla, h)$ is 1-conformally flat if and only if the dual statistical manifold $(N, \nabla', h)$ is $(-1)$-conformally flat [4].

Henceforth, we suppose that $g$ is positive definite.

Let $\tilde{E}$ be the gradient vector field of $\varphi$ on $\Omega$ defined by

$$g(X, \tilde{E}) = d\varphi(X) \quad \text{for } X \in T\Omega,$$

where $T\Omega$ is the set of all tangent vector fields on $\Omega$. We set

$$E = -d\varphi(\tilde{E})^{-1}\tilde{E} \quad \text{on} \quad \Omega_o = \{p \in \Omega \mid d\varphi_p \neq 0\}.$$

For $p \in \Omega_o$, $E_p$ is perpendicular to $T_pM$ with respect to $g$, where $M \subset \Omega_o$ is a level surface of $\varphi$ containing $p$ and $T_pM$ is the set of all tangent vectors at $p$ on $M$.

Let $x$ be a canonical immersion of an $n$-dimensional level surface $M$ into $\Omega$. For $D$ and an affine immersion $(x, E)$, we can define the induced affine connection $D^E$, the affine fundamental form $g^E$ on $M$ by

$$D_X Y = D^E_X Y + g^E(X,Y)E \quad \text{for } X, Y \in TM.$$

We denote by $D^M$ and $g^M$ the connection and the Riemannian metric on $M$ induced by $D$ and $g$. A submanifold being a statistical manifold is called a statistical submanifold. Then the triple $(M, D^M, g^M)$ is the statistical submanifold realized in $(\Omega, D, g)$, which coincides with the manifold $(M, D^E, g^E)$ induced by an affine immersion $(x, E)$. This fact leads the next theorem.

**Theorem 2.1** *([9]) Let $M$ be a simply connected $n$-dimensional level surface of $\varphi$ on an $(n+1)$-dimensional Hessian domain $(\Omega, D, g = Dd\varphi)$ with a Riemannian metric $g$, and suppose that $n \geq 2$. If we consider $(\Omega, D, g)$ a flat statistical manifold, $(M, D^M, g^M)$ is a 1-conformally flat statistical submanifold of $(\Omega, D, g)$, where we denote by $D^M$ and $g^M$ the connection and the Riemannian metric on $M$ induced by $D$ and $g$.*

Conversely, on realization of a 1-conformally flat statistical manifold we have:

**Theorem 2.2** *([9]) An arbitrary 1-conformally flat statistical manifold of* $\dim n \geq 2$ *with a Riemannian metric can be locally realized as a submanifold of a flat statistical manifold of* $\dim\,(n+1)$.

# 3 Necessity of the conditions

In this section we show necessary conditions for a Hessian domain. For this purpose we describe properties of affine immersions.

Let $(\Omega, D, g = Dd\varphi)$ be an $(n+1)$-dimensional Hessian domain, and $(M, D^M, g^M)$ an $n$-dimensional 1-conformally flat statistical submanifold by a level surface $M$ of $\varphi$ with $n \geq 2$. We set $\mathcal{F}$ is a foliation on $\Omega_o$ by level surfaces of $\varphi$.

Let $(x^M, E^M)$ be an immersion realizing $(M, D^M, g^M)$ in $\mathbf{A}^{n+1}$. We set $\iota^M$ is the conormal immersion for $x^M$, i.e., denoting by $\langle a, b\rangle$ a pairing of $a \in \mathbf{A}^*_{n+1}$ and $b \in \mathbf{A}^{n+1}$,

$$\langle \iota^M(p), Y_p\rangle = 0 \ \text{ for } Y_p \in T_pM, \quad \langle \iota^M(p), E^M_p\rangle = 1$$

for $p \in M$, considering $T_p\mathbf{A}^{n+1}$ with $\mathbf{A}^{n+1}$. The immersion $\iota^M$ satisfies that

$$\langle \iota^M_*(Y), E^M\rangle = 0, \quad \langle \iota^M_*(Y), X\rangle = -g^M(Y, X) \ \text{ for } X, Y \in TM.$$

It is also known that the conormal immersion $\iota^M$ coincides with the gradient mapping $\iota : \Omega \to \Omega^* = \iota(\Omega)$ on $M$ [9]. Moreover an immersion $(x^M, E^M)$ is equiaffine, i.e.,

$$D_X E^M = S^{E^M}(X) \in TM \ \text{ for } X \in TM$$

(We call $S^{E^M}$ the shape operator.) [5] [6]. With notations defined in this section we can also describe

$$D_X Y = D^M_X Y + g^M(X, Y)E^M \quad \text{for } X, Y \in TM.$$

Then the next lemma holds.

**Lemma 3.1** *An $(n+1)$-dimensional Hessian domain $(\Omega, D, g = Dd\varphi)$ and $n$-dimensional level surfaces $\{(M, D^M, g^M)|M \in \mathcal{F}\}$ $(n \geq 2)$ satisfy the following conditions:*
*(i) a mapping $E : \Omega_o \to \mathbf{A}^{n+1}$ defined by $E(p) = E^M(p)$ for $p \in M$ is differenciable;*
*(ii) the gradient mapping $\iota : \Omega \to \Omega^* = \iota(\Omega) \subset \mathbf{A}^*_{n+1}$ is a diffeomorphism;*
*(iii) $X(g(E, E)) = -g(D_E E, X)$ for $X \in TM$;*
*(iv) $S^{E^M}(X) = -(d\lambda(E) + g(E, E))(X)$ for $X \in TM$, where $\lambda$ is a function on a small open set $U \subset \Omega_o$ such that $e^{\lambda(\hat{p})}\iota(p) = \iota(\hat{p})$, $\hat{p} \in U$ for $p \in U \cap M$;*
*(v) $D_E X = 0$ for $X \in T\Omega_o$, where $X_p = X_{\hat{p}} \in TM_p = TM_{\hat{p}}$ if $e^{\lambda(\hat{p})}\iota(p) = \iota(\hat{p})$.*

*Proof.* The conditions (i) and (ii) are clear.

For the proof of (iii), we calculate $(D_X g)(E,E)$ and $(D_E g)(E,X)$ for $X \in TM$. By $D_X E^M = S^{E^M}(X) \in TM$, it holds that

$$(D_X g)(E,E) = X(g(E,E)) - 2g(D_X E, E) = X(g(E,E)).$$

By the definitions of the gradient vector field $\tilde{E}$ for $g$ and the conormal vector field $E = -d\varphi(\tilde{E})^{-1}\tilde{E}$, we have

$$\begin{aligned}
(D_E g)(E,X) &= E(g(E,X)) - g(D_E E, X) - g(E, D_E X) && (1)\\
&= -g(D_E E, X) - d\varphi(\tilde{E})^{-2} d\varphi(D_{\tilde{E}} X) && (2)\\
&= -g(D_E E, X) - d\varphi(\tilde{E})^{-2}(\tilde{E}(d\varphi(X)) - (D_{\tilde{E}} d\varphi)(X)) && (3)\\
&= -g(D_E E, X). && (4)
\end{aligned}$$

In the above we also make use of $d\varphi(X) = 0$ and $(D_{\tilde{E}} d\varphi)(X) = g(\tilde{E}, X) = 0$. From the Codazzi equation for $(D, g)$ on a Hessian domain, it follows that

$$(D_X g)(E,E) = (D_E g)(E,X)$$

[8]. Thus we obtain the condition (iii).

For the proofs of (iv) and (v), we calculate $(D_X g)(E,Z)$ and $(D_E g)(X,Z)$ for $X, Z \in TM$. It holds that

$$\begin{aligned}
(D_X g)(E,Z) &= X(g(E,Z)) - g(D_X E, Z) - g(E, g^M(X,Z)E)\\
&= -g^M(S^{E^M}(X), Z) - g(E,E)g^M(X,Z).
\end{aligned}$$

Conormal immersions $\{(\iota^M, g^M) | M \in \mathcal{F}\}$ are projectively equivalent and conformally equivalent, and it holds that $g^{\hat{M}} = e^{\lambda} g^M$ for each $M, \hat{M} \in \mathcal{F}$, where $\lambda$ is a function on a small open set $U \subset N$ ($M \cap U, \hat{M} \cap U \neq \phi$) [6]. Hence the next follows; for $p \in M$,

$$\begin{aligned}
E(g(X,Z))|_p &= E(e^{\lambda} g^M(X,Z))|_p = (Ee^{\lambda})|_p g^M(X,Z)\\
&= (E\lambda)|_p e^{\lambda(p)} g^M(X,Z) = d\lambda(E)|_p g^M(X,Z).
\end{aligned}$$

In addition, it holds that $g(E, D_E X) = 0$ by the calculation of eq.(1) to eq.(4). Then we have the condition (v), i.e., $D_E X = 0$. Thus it holds that

$$\begin{aligned}
(D_E g)(X,Z) &= E(g(X,Z)) - g(D_E X, Z) - g(X, D_E Z)\\
&= d\lambda(E)|_p g^M(X,Z).
\end{aligned}$$

From the Codazzi equation for $(D, g)$, it follows that

$$(D_X g)(E,Z) = (D_E g)(X,Z).$$

Thus $(M, D^M, g^M)$ satisfies the condition (iv). ♢

**Remark 3.2** Hao and Shima calculated $(D_X g)(\tilde{E}, \tilde{E})$ and $(D_{\tilde{E}} g)(\tilde{E}, X)$ for $(x, \tilde{E})$, not for $(x, E)$, and showed that the transversal connection form $\tau^{\tilde{E}}$ vanishes if and only if $D_{\tilde{E}}\tilde{E} = \mu\tilde{E}$ [2][8]. We proved Lemma 3.1 (iii) with their technique.

**Remark 3.3** By Lemma 3.1 (iii), $g(E, E)$ is constant on each $M$ if and only if $D_E E = \mu E$ for $\mu \in \mathbf{R}$ on $\Omega_o$.

# 4 Foliations constructed by 1-conformally flat statistical manifolds

Let $\mathcal{F}$ be a foliation on a differentiable manifold $N$ of dimension $n \geq 2$ and codimension 1, and the triple $(M, \nabla^M, h^M)$ a 1-conformally flat statistical manifold for each leaf $M \in \mathcal{F}$. Suppose that a non-degenerate affine immersion $(x^M, E^M)$ realizes $(M, \nabla^M, h^M)$ in $\mathbf{A}^{n+1}$, and that a mapping $x : N \to \Omega$ defined by $x(p) = x^M(p)$ for $p \in M$ is a diffeomorphism, where $\Omega = \cup_{M\in\mathcal{F}} x^M(M) \subset \mathbf{A}^{n+1}$ is a domain diffeomorphic to $N$.

We set $\iota^M$ is the conormal immersion for $x^M$, and $S^{E^M}$ the shape operator of $(x^M, E^M)$. With notations defined in this section we can describe

$$D_X Y = \nabla^M_X Y + h^M(X, Y)E^M \quad \text{for } X, Y \in TM.$$

Then the next main theorem holds.

**Theorem 4.1** *(main theorem) Each 1-conformally flat statistical leaf* $(M, \nabla^M, h^M)$ *of* $\mathcal{F}$ *is locally realized as a level surface of the common Hessian domain and* $N = \cup_{M\in\mathcal{F}} M$ *is locally diffeomorphic to the Hessian domain, if and only if a foliation* $\mathcal{F}$ *satisfies the following conditions:*
*(i) a mapping* $E : N \to \mathbf{A}^{n+1}$ *defined by* $E(p) = E^M(p)$ *for* $p \in M$ *is differenciable;*
*(ii) a mapping* $\iota : N \to \Omega^*$ *defined by* $\iota(p) = \iota^M(p)$ *for* $p \in M$ *is a diffeomorphism, where* $\Omega^* = \cup_{M\in\mathcal{F}} \iota^M(M) \subset \mathbf{A}^*_{n+1}$*;*
*(iii) there exists a function* $\rho : M \to \mathbf{R}^+$ *such that*

$$\begin{aligned} X\rho &= -h^M((D_E E)|_{TM}, X), & (5)\\ S^{E^M}(X) &= -(d\lambda(E) + \rho)(X) \quad \textit{for } X \in TM, & (6) \end{aligned}$$

*where we set* $D_E E = (D_E E)|_{TM} + \mu E^M$, $\mu \in \mathbf{R}$, *and where* $\lambda$ *is a function on a small open set* $U \subset N$ *such that* $e^{\lambda(\hat{p})}\iota(p) = \iota(\hat{p})$, $\hat{p} \in U$ *for* $p \in U \cap M$*;*
*(iv)* $D_E X = 0$ *for* $X \in T\Omega$, *where* $X_p = X_{\hat{p}} \in T_p M = T_{\hat{p}} M$ *if* $e^{\lambda(\hat{p})}\iota(p) = \iota(\hat{p})$.

*Proof.* We consider a manifold $N$ a domain $\Omega \subset \mathbf{A}^{n+1}$, and define a metric $g$ on $\Omega$ by

$$g(Y,X) = h^M(Y,X),\ \ g(E,E) = \rho,\ \ g(Y,E) = 0 \ \text{ for } X, Y \in TM \subset T\Omega.$$

Let us prove that $(D, g)$ satisfies the Codazzi equation

$$(D_X g)(Y,Z) = (D_Y g)(X,Z) \ \text{ for all } \ X, Y \text{and } Z \in T\Omega.$$

In the case of $X, Y$ and $Z \in TM$, we have

$$\begin{aligned}(D_X g)(Y,Z) &= X(g(Y,Z)) - g(D_X Y, Z) - g(Y, D_X Z)\\ &= X(h^M(Y,Z)) - g(\nabla^M_X Y, Z) - g(Y, \nabla^M_X Z)\\ &= (\nabla^M_X h^M)(Y,Z).\end{aligned}$$

Similarly it holds that

$$(D_Y g)(X,Z) = (\nabla^M_Y h^M)(X,Z).$$

Recall the Codazzi equation for an equiaffine immersion $(x^M, E^M)$;

$$(\nabla^M_X h^M)(Y,Z) = (\nabla^M_Y h^M)(X,Z)$$

[6]. Then we have the Codazzi equation for $(D, g)$.

In the case of $X, Y \in TM$ and E on $M$, we have

$$\begin{aligned}(D_X g)(Y,E) &= X(g(Y,E)) - g(h^M(X,Y)E, E) - g(Y, D_X E)\\ &= -\rho h^M(X,Y) - h^M(Y, S^{E^M}(X)).\end{aligned}$$

Similarly it holds that

$$(D_Y g)(X,E) = -\rho h^M(X,Y) - h^M(X, S^{E^M}(Y)).$$

Recall the Ricci equation for an equiaffine immersion $(x^M, E^M)$;

$$h^M(S^{E^M}(X), Y) = h^M(X, S^{E^M}(Y))$$

[6]. Then we have the Codazzi equation

$$(D_X g)(Y,E) = (D_Y g)(X,E).$$

In the case of $X, Z \in TM$ and E on $M$, similarly we have

$$(D_X g)(E,Z) = -\rho h^M(X,Z) - h^M(S^{E^M}(X), Z).$$

By the condition (ii), conormal immersions $\{(\iota^M, h^M) | M \in \mathcal{F}\}$ are projectively equivalent and conformally equivalent. Then it holds that $h^{\hat{M}}|_{\hat{p}} = e^{\lambda(\hat{p})} h^M|_p$ for $e^{\lambda(\hat{p})}\iota(p) = \iota(\hat{p})$ locally [6]. Hence the next follows; for $p \in M$,

$$\begin{aligned} E(g(X,Z))|_p &= E(e^{\lambda} h^M(X,Z))|_p = (Ee^{\lambda})|_p h^M(X,Z) \\ &= (E\lambda)|_p e^{\lambda(p)} h^M(X,Z) = d\lambda(E)|_p h^M(X,Z). \end{aligned}$$

Thus it holds that

$$\begin{aligned} (D_E g)(X,Z) &= E(g(X,Z)) - g(D_E X, Z) - g(X, D_E Z) \\ &= d\lambda(E)|_p h^M(X,Z) \end{aligned}$$

from the condition (iv). By eq.(6), we have the Codazzi equation

$$(D_X g)(E,Z) = (D_E g)(X,Z).$$

In the case of $X \in TM$ and E on $M$, we have

$$\begin{aligned} (D_X g)(E,E) &= X(g(E,E)) - g(g(X,E)E,E) - g(E, g(X,E)E) \\ &= X(g(E,E)) = X\rho. \end{aligned}$$

Moreover it holds that

$$\begin{aligned} (D_E g)(X,E) &= E(g(X,E)) - g(D_E X, E) - g(X, D_E E) \\ &= -h^M((D_E E)|_{TM}, X). \end{aligned}$$

By eq.(5), we have the Codazzi equation

$$(D_X g)(E,E) = (D_E g)(X,E).$$

In the case of $X = Y = E$ and $Z \in T\Omega$, clearly we have

$$(D_X g)(Y,Z) = (D_Y g)(X,Z) = (D_E g)(E,Z).$$

Hence $(D, g)$ satisfies the Codazzi equation. Thus $g$ is a Hessian metric by Proposition 2.1 on [8]. By the definition of $g$ we can consider that each leaf $(M, \nabla^M, h^M)$ of $\mathcal{F}$ is a level surface of the Hessian domain $(\Omega, D, g)$.

Necessity of the conditions (i) to (iv) follows from Lemma 3.1. $\diamondsuit$

Last we talk about a projectively flat connection and a dual-projectively flat connection. Kurose and Ivanov proved the next theorems, respectively.

**Theorem 4.2** *([4]) A statistical manifold $(N, \nabla, h)$ is 1-conformally flat if and only if the dual connection $\nabla'$ is a projectively flat connection with symmetric Ricci tensor.*

**Theorem 4.3** *([3]) A statistical manifold* $(N, \nabla, h)$ *is* 1*-conformally flat if and only if* $\nabla$ *is a dual-projectively flat connection with symmetric Ricci tensor.*

Thus we can describe Theorem 4.1 as the next.

**Corollary 4.4** *Let* $\nabla^M$ *be a dual-projectively flat connection with symmetric Ricci tensor for each* $M \in \mathcal{F}$*. Then each statistical leaf* $(M, \nabla^M, h^M)$ *of a foliation* $\mathcal{F}$ *is locally realized as a level surface of the common Hessian domain and* $N = \cup_{M \in \mathcal{F}} M$ *is locally diffeomorphic to the Hessian domain if and only if* $\mathcal{F}$ *satisfies the conditions (i) to (iv) of Theorem 4.1.*

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